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THE APPLICATION OF HARMONIC POLYNOMIALS TO THE SOLUTION OF DIRICHLET PROBLEM FOR THE CIRCULAR DOMAIN

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Abstract: The subject of the paper is the construction of the solution $u(x, y)$ to the Laplace equation in circular domain.

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1. Introduction

The subject of the paper is the construction of the solution $u(x, y)$ to the Laplace equation

$$\Delta u(x, y) = 0, \tag{1}$$

in the circular domain

$$D = \{(x, y) : \ x^2 + y^2 < R^2\}$$

satisfying the boundary-value condition

$$u(x, y) = f(x, y) , \qquad (x, y) \in B(D). \tag{2}$$

Applying the harmonic polynomials $\{P_n(x, y)\}$ and their linear combinations

$$W_n(x, y) = C_0P_0(x, y) + C_1P_1(x, y) + \cdots + C_nP_n(x, y) \tag{3}$$

$(C_0, \ldots , C_n)$ is a system of arbitrary real constants, by Arzela and Harnack

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theorems we shall construct the solution of the problem (1), (2) using a method, which is new in literature.

2. Harmonic Polynomials $P_n(x, y)$

Consider the sequence of harmonic polynomials

$$P_0(x,y) = 1,$$

$$\begin{aligned} P_1(x,y) = \mathrm{Real} z = x,\ P_2(x,y) = \mathrm{Im} z = y, \\ P_3(x,y) = \mathrm{Real} z^2 = x^2 - y^2, \\ P_4(x,y) = \mathrm{Im} z^2 = 2xy, \end{aligned} \tag{4}$$

$$P_{2n-1}(x,y) = \mathrm{Real} z^n = x^n - \binom{n}{2} x^{n-2} y^2 + \cdots ,$$

$$P_{2n}(x,y) = \mathrm{Im} z^n = -\binom{n}{1} x^{n-1} y + \binom{n}{3} x^{n-3} y^3 + \cdots . \tag{5}$$

Hence we obtain the sequence of polynomials

$$P_0(x,y) = 1,\ P_1^1(x,y) = x,\ P_2^1(x,y) = y, \ldots, P_{2n-1}^1(x,y) = \mathrm{Real} z^n, \tag{6}$$

$$P_{2n}^1(x,y) = \mathrm{Jm} z^n, \ldots,$$

or

$$P_0, P_1, P_2, P_3, P_4, \ldots, P_{n-1}, P_n, \ldots \tag{7}$$

Lemma 1. *The system (7) of the polynomials $\{P_n(x,y)\}$ is linearly independent.*

Proof. Let n be even or odd. Let $(C_0, C_1, \ldots, C_{n-1}, C_n)$ denote arbitrary constants. From the identity

$$\begin{aligned} &W_n(x, y, C_0, C_1, \ldots, C_{n-1}, C_n) \\ &\quad = C_0 + C_1 x + C_2 y + \cdots + C_{n-1} P_{n-1}(x,y) + C_n P_n(x,y) = 0\,, \end{aligned} \tag{8}$$

we have that

$$C_0 = C_1 = C_2 = \cdots = C_{n-1} = C_n = 0. \tag{9}$$

Indeed, the polynomials $P_n(x,y)$ are of different degree with respect to x and y. Differentiating with respect to x or y in the point $(0,0)$ we obtain

$C_0 \cdot 1 = 0$, $C_0 = 0$, $C_1 \cdot 1 = 0$, $C_1 = 0$. Differentiating the identity (8) with respect to y we obtain $C_2 = 0$ and so on.

Finally we obtain (9).

Consider the sequence

$$\{S_n\} = \{(x_0, y_0), \ldots, (x_n, y_n)\} \in B(D)$$

and the family of sequences (S_n) such that $S_n \subset S_{n+1}$, the set $\sum_{n=0}^{\infty} S_n$ is everywhere dense in $B(D)$.

Consider the system of the linear equations for n odd or n even with unknown C_i, $i = 0, 1, \ldots, n$ for $(x_0, t_0), \ldots, (x_n, y_n) \in B(D)$,

$$\begin{aligned} W_n(x_0, y_0) &= C_1 P_1(x_0, y_0) + \cdots + C_n P_n(x_0, y_0) = f(x_0, y_0)\,, \\ &\vdots \\ W_n(x_n, y_n) &= C_0 P_0(x_n, y_n) + C_1 P_1(x_n, y_n) + \cdots + C_n P_n(x_n, y_n) \\ &= f(x_n, y_n). \end{aligned} \tag{10}$$

Since the polynomials $P_i(x, y)$ are linearly independent, by Cramers formula we obtain the solution of the system (10)

$$\begin{aligned} C_i &= \frac{D_i}{D^1}, \quad i = 0, 1, \ldots, n\,, \\ D^1 &= \det[P_i(x_k, y_k)], \quad i, k = 0, 1, \ldots, n\,, \end{aligned} \tag{11}$$

$$D_i = \det \begin{bmatrix} P_0(x_0, y_0), \ldots, P_{i-1}(x_0, y_0), & f(x_0, y_0), P_{i+1}(x_0, y_0), \ldots, P_n(x_0, y_0) \\ \ldots & \ldots \\ P_0(x_n, y_n), \ldots, P_{i-1}(x_n, y_n), & f(x_n, y_n), P_{i+1}(x_n, y_n), \ldots, P_n(x_n, y_n) \end{bmatrix},$$

where $i = 0, 1, \ldots, n$. Considder the sequence

$$W_n(x, y, C_0, \ldots, C_n, f) = \{C_0 P_0(x, y) + \cdots + C_n P_n(x, y)\}\,,$$

for which

$$C_0 P_0(x, y) + \cdots + C_n P_n(x, y) = f(x, y), (x, y) \in B(D)\,.$$

Consider the family

$$H = W_n(x, y, C, f), \quad C = (C_0, \ldots, C_n)\,.$$

3. Application of the Polynomials W_n to the Solution of the Problem (1), (2)

Consider the polynomials $\{W_n(x,y,C,f)\}$, $n = 0,1,\dots$

In the sequel we suppose that the function $f(x,y)$ satisfies the Lipschitz condition, i.e. for all pairs $(x_i,y_i),(x_j,y_j) \in B(D)$, $i=1$, $j=2$

$$|f(x_1,y_1) - f(x_2,y_2)| \le q(|x_1 - x_2| + |y_1 - y_2|).$$

By [2] the maximum principle

$$\sup_D W_n(x,y,C,f) < \sup_{B(D)} W_n(x,y,C,f) < M = \sup f\,,$$

$$\inf_D W_n(x,y,C,f) > \inf_{B(D)} W_n(x,y,C,f) > m = \inf f$$

holds.

Lemma 2. *The functions $\{W_n(x,y,C,f)\}$ are 1^0 equibounded, 2^0 equcontinuous.*

Proof. Ad 1^0 By maximum principle we obtain equiboundness.

Ad 2^0 By Lipschitz condition we obtain for $(x_1,y_1),(x_2,y_2) \in B(D)$

$$|W_n(x_1,y_1) - W_n(x_2,y_2)| \\ = |f(x_1,y_1) - f(x_2,y_2)| \le q(|x_1,x_2| + |y_1 - y_2|)\,.$$

Hence we obtain equicontinuity in $D \cup B(D)$. □

4. Arzela and Harnack Theorems

Theorem 1. *If the functions of the sequences $\{W_n(x,y,C,f)\}$ are equibounded and equicontinuous, uniformly convergent on a compact set $B(D)$, thus by Arzela Theorem we can choose from $\{W_n(x,y,f)\}$ the sequence $\overline{W}_n(x,y,f)$ uniformly convergent in $D \cup B(D)$ and apply Harnack Theorem.*

Harnack Theorem. *Let*

$$W(x,y,C,f) = \lim_{n\to\infty} \overline{W}_n(x,y,C,f).$$

Then $W(x,y,C,f)$ is harmonic in D and satisfies boundary-value condition (2).

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