International Journal of Pure and Applied Mathematics
Volume 16 No. 1 2004, 63-71

TRIPLE POINTS AND THE WEAK NON-DEFECTIVITY OF VERONESE EMBEDDINGS OF PROJECTIVE SPACES

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Abstract: Fix integers $n \geq 2$, $d \geq 4$ and $k \geq 0$ such that

$$k(n+1) + (n+2)(n+1)/2 \leq \binom{n+d}{n}$$

and $k+1$ general points $P_1, \dots, P_{k+1} \in \mathbf{P}^n$. Let $\Sigma(P_1, \dots, P_{k+1})$ (resp. $\Sigma(P_1, \dots, P_{k+1})'$) denote the projective space of all degree d hypersurfaces of $\mathbf{P}^n$ singular at each point P_i, $1 \leq i \leq k+1$, (resp. singular at each P_i and with a triple point at P_{k+1}). Here we use Horace Method to prove that $\dim(\Sigma(P_1, \dots, P_{k+1})') = \binom{n+d}{n} - (n+1)k - (n+2)(n+1)/2 - 1$, $\dim(\Sigma(P_1, \dots, P_{k+1})) = \binom{n+d}{n} - (k+1)(n+1) - 1$ and (in characteristic zero) a general $F \in \Sigma(P_1, \dots, P_{k+1})$ satisfies $\mathrm{Sing}(F) = \{P_1, \dots, P_{k+1}\}$ and it has an ordinary node at each P_i. This result implies that the order d Veronese embedding of $\mathbf{P}^n$ is not weakly k-defective in the sense of Ciliberto and Chiantini.

AMS Subject Classification: 14N05
Key Words: Veronese variety, weakly defective variety, zero-dimensional scheme, double point, fat point, Veronese embedding

1. Introduction

The main aim of this paper is to use the so-called Horace Method introduced by A. Hirschowitz to prove the following result.

Received: July 5, 2004

Theorem 1. *Fix integers $n \geq 2$, $d \geq 4$ and $k \geq 0$ such that*

$$k(n+1) + (n+2)(n+1)/2 \leq \binom{n+d}{n} \tag{1}$$

and a general $S \subset \mathbf{P}^n$ such that $\mathrm{card}(S) = k+1$. *Let $\Sigma(S)$ denote the projective space of all degree d hypersurfaces of $\mathbf{P}^n$ singular at each point of S. Then* $\dim(\Sigma(S)) = \binom{n+d}{n} - (k+1)(n+1) - 1$. *A general $F \in \Sigma(S)$ satisfies* $\mathrm{Sing}(F) = S$ *and it has an ordinary node at each point of S.*

This result implies that the order d Veronese embedding of $\mathbf{P}^n$ is not weakly k-defective in the sense of Ciliberto and Chiantini (see [8]). More precisely, the well-known equality $\dim(\Sigma(S)) = \binom{n+d}{n} - (k+1)(n+1) - 1$ (proved for more triples (n, d, k) by J. Alexander and A. Hirschowitz ([1], [2], [3], [4], [7])) means that the order d Veronese embedding of $\mathbf{P}^n$ is not k-defective by Terracini Lemma ([9]), while the additional assertion that $\mathrm{Sing}(F)$ has no positive-dimensional part intersecting S means that this embedding is not weakly k-defective. For related examples in which the singular locus has positive dimension, see [10] and [9], Remark 6.2, which quotes [11].

Theorem 1 will be a very easy consequence of [8], Theorem 1.4, and the following result.

Theorem 2. *Fix integers $n \geq 2$, $d \geq 4$ and $k \geq 0$ such that the inequality (1) is satified. Fix $k+1$ general points P_i, $0 \leq i \leq k$, and call Σ the projective space of all degree d hypersurfaces of $\mathbf{P}^n$ singular at each point $P_1, \dots, P_k$ and with multiplicity at least 3 at P_0. Then* $\dim(\Sigma) = \binom{n+d}{n} - k(n+1) - (n+2)(n+1)/2 - 1$.

Similar results should be sistematically checked for other projective varieties embedded in a fix projective space to check that the embedding is not weakly k-defective (see Lemma 1).

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Our proof of Theorem 2 will be characteristic free, while our proof of Theorem 1 depends heavily from the characteristic zero assumption: a key tool will be [8], Theorem 1.4.

2. The Proofs

For any integer $m > 0$ and any $P \in \mathbf{P}^n$ let mP denote the infinitesimal neighborhood of order $m-1$ of P in $\mathbf{P}^n$, i.e. the closed zero-dimensional subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_P)^m$ as its ideal sheaf. We have $\mathrm{length}(mP) = \binom{n+m-1}{n}$. We will say that $2P$ (resp. $3P$) is the double (resp. triple) point with P as its support. Notice that $\text{length}(3P) = (n+2)(n+1)/2$ and $\text{length}(2P) = n+1$. These equalities explain the formulas for $\dim(\Sigma(S))$ and $\dim(\Sigma)$ in the statements of Theorem 1 and Theorem 2. For any closed subscheme $Z \subseteq \mathbf{P}^n$ and any integer $d \geq 1$ let $\rho_{Z,n,d} : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \to H^0(Z, \mathcal{O}_Z(d))$ denote the restriction map.

We will sometimes use the notation $2P$ and $3P$ when P is a smooth point of an arbitrary variety. With this convention we can state the following result.

Lemma 1. *Let X be an n-dimensional integral projective variety and Φ a very ample linear system on X. Fix $k+1$ general points $P_i \in X$, $0 \leq i \leq k$. Set $\Phi(-2P_0 - 2P_1 - \cdots - 2P_k) := \{F \in \Phi : F$ is singular at each point P_i, $0 \leq i \leq k\}$ and $\Phi(-3P_0 - 2P_1 - \cdots - 2P_k) := \{F \in \Phi(-2P_0 - 2P_1 - \cdots - 2P_k) : F$ has multiplicity at least 3 at P_0. Assume $\dim(\Phi(-3P_0 - 2P_1 - \cdots - 2P_k)) = \dim(\Phi) - (n+2)(n+1)/2 - k(n+1)$. Then $\dim(\Phi(-2P_0 - 2P_1 - \cdots - 2P_k)) = \dim(\Phi) - (k+1)(n+1)$ and a general $F \in \Phi(-2P_0 - 2P_1 - \cdots - 2P_k)$ satisfies $\text{Sing}(F) = \{P_0, \ldots, P_k\}$ and it has an ordinary node at each P_i, $0 \leq i \leq k$.*

Proof. For all zero-dimensional schemes $W \subseteq Z$ the quotient map $\mathcal{O}_Z \to \mathcal{O}_W$ induces a surjective map $H^0(Z, \mathcal{O}_Z) \to H^0(W, \mathcal{O}_W)$. This observation and the assumption on $\dim(\Phi(-3P_0 - 2P_1 - \cdots - 2P_k))$ obviously give the formula for $\dim(\Phi(-2P_0 - 2P_1 - \cdots - 2P_k))$. Set $\Phi(2P_1 - \cdots - 2P_k) := \{F \in \Phi : F$ is singular at each point P_j, $1 \leq j \leq k$ and call $\Psi(-2P_1 - \cdots - 2P_k)$ the corresponding vector space. The assumption on $\dim(\Phi(-3P_0 - 2P_1 - \cdots - 2P_k))$ implies that the restriction map $\Psi(2P_1 - \cdots - 2P_k) \to H^0(3P, \mathcal{O}_{3P})$ is surjective. Hence a general $F \in \Phi(-2P_0 - 2P_1 - \cdots - 2P_k)$ has a non-degenerate quadratic form as the first term of its Taylor expansion (with respect to any formal parameters at P_0). Any such hypersurface has an isolated singular point at P_0 and this singularity is an ordinary quadratic point. Apply [8], Theorem 1.4. □

For all integers $n \geq 2$ and $d \geq 3$ we define the integers $x_{n,d}$ and $y_{n,d}$ using the following relations:

$$(n+1)x_{n,d} + \binom{n+2}{2} + y_{n,d} = \binom{n+d}{n}, \; 0 \leq y_{n,d} \leq n\,. \tag{2}$$

If $y_{n,d} = 0$, then set $x'_{n,d} := x_{n,d}$ and $y'_{n,d} := 0$. If $y_{n,d} > 0$, then set $x'_{n,d} := x_{n,d} + 1$ and $y'_{n,d} := n + 1 - y_{n,d}$. Hence if $y_{n,d} > 0$ we have the relations

$$(n+1)x'_{n,d} + \binom{n+2}{2} - y'_{n,d} = \binom{n+d}{n}, \; 1 \leq y'_{n,d} \leq n\,, \tag{3}$$

which uniquely determine the integers $x'_{n,d}$ and $y'_{n,d}$.

Definition 1. For all integers $n \geq 2$ and $d \geq 3$ we define the assertions $A_{n,d}$ and $A'_{n,d}$ in the following way:

$A_{n,d}$: For a general union $Z \subset \mathbf{P}^n$ of one triple point and $x_{n,d}$ double points the restriction map $\rho_{Z,n,d}$ is surjective.

$A'_{n,d}$: For a general union $W \subset \mathbf{P}^n$ of one triple point and $x'_{n,d}$ double points the restriction map $\rho_{W,n,d}$ is injective.

In the stament of $A_{n,d}$ the map $\rho_{Z,n,d}$ is surjective if and only if $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ if and only if $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = y_{n,d}$. In the statement of $A'_{n,d}$ the map $\rho_{W,n,d}$ is injective if and only if $h^0(\mathbf{P}^n, \mathcal{I}_W(d)) = 0$ if and only if $h^1(\mathbf{P}^n, \mathcal{I}_W(d)) = y'_{n,d}$. To check for what integers n, d assertions $A_{n,d}$ and/or $A'_{n,d}$ are true will be the main step in the proof of Theorem 2. Hence the main aim of this paper is to prove the following result.

Lemma 2. *Fix integers $n \geq 2$ and $d \geq 3$. $A_{n,d}$ is true if and only if $d \neq 3$. $A'_{n,d}$ is true if and only if $d \neq 3$.*

Remark 1. Fix integers $n \geq 2$, $k \geq 0$ and $d \geq 3$. Let $Z_{n,k} \subset \mathbf{P}^n$ be a general union of a triple point and k double points. If $A_{n,d}$ is true and $k \leq x_{n,d}$, then $h^1(\mathbf{P}^n, \mathcal{I}_{Z_{n,k}}(d)) = 0$. If $A'_{n,d}$ is satisfied and $k \geq x'_{n,d}$, then $h^0(\mathbf{P}^n, \mathcal{I}_{Z_{n,k}}(d)) = 0$.

Remark 2. Fix $P \in \mathbf{P}^n$, $n \geq 2$, and a hyperplane $H \subset \mathbf{P}^n$ such that $P \notin H$. Let $h_P : \mathbf{P}^n \backslash \{P\} \to H$ denote the linear projection from P. Let $S \subset \mathbf{P}^n$ be a degree three hypersurface. S has multiplicity three at P if and only if it a cone with vertex containing P. Hence the linear projection h_P induces an isomorphism $H^0(\mathbf{P}^n, \mathcal{I}_{3P}(3)) \cong H^0(H, \mathcal{O}_H(3))$. A double point of $\mathbf{P}^n$ has length $n+1$, while a double point of H has length n. For any integer $k \geq 0$ and any k distinct points $P_1, \dots, P_k \in \mathbf{P}^n \backslash \{P\}$ we have $h^0(\mathbf{P}^n, \mathcal{I}_{3P \cup 2P_1 \cup \dots \cup 2P_k}(3)) = h^0(H, \mathcal{I}_{2h_P(P_1) \cup \dots \cup 2h_P(P_k)}(3))$. Hence, roughly speaking, each double point may impose at most n independent conditions to any linear subsystem of $|\mathcal{I}_{3P}(3)|$. In particular we easily see that $A_{n,3}$ and $A'_{n,3}$ are false. If the points $P_1, \dots, P_k$ are general, then the points $h_P(P_1), \dots, h_P(P_k)$ are general in H. Hence from the known cohomology of a general union of double points in $\mathbf{P}^{n-1}$ (see [1], [2], [3], [4], [7]) we obtain the following table of exceptional cases.

Proposition 1. *For all integers $n \geq 2$ and $k \geq 0$ let $Z_{n,k} \subset \mathbf{P}^n$ be a general union of a triple point and k double points. We have $h^0(\mathbf{P}^n, \mathcal{I}_{Z_{n,k}}(3)) = 0$ if and only if $k \geq \ulcorner (n+2)(n+1)/6 \urcorner$. If $k < \ulcorner (n+2)(n+1)/6 \urcorner$, then $h^1(\mathbf{P}^n, \mathcal{I}_{Z_{n,k}}(3)) = k$.*

We will often use the following elementary form of the so-called Horace Lemma.

Lemma 3. *Let $H \subset \mathbf{P}^n$ be a hyperplane and $Z \subset \mathbf{P}^n$ a closed subscheme. Then:*

(a) $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)}}(d-1)) + h^0(H, \mathcal{I}_{Z\cap H}(d))$.

(b) $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)}}(d-1)) + h^1(H, \mathcal{I}_{Z\cap H}(d))$.

Proof. By the very definition of residual scheme with respect to H, there is the following exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(Z)}}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\cap H}(d) \to 0\,, \tag{4}$$

whose long cohomology exact sequence proves the lemma. □

The following result is a very particular case of [5], Lemma 2.3 (see in particular Figure 1 at p. 308); the case $n = 2$ was used in [6] in the same way and called there the double residue trick; alternatively, use [7], Lemma 5.

Lemma 4. *Let $H \subset \mathbf{P}^n$ be a hyperplane, $Z \subset \mathbf{P}^n$ a closed subscheme not containing H and s a positive integer. Let U be the union of Z and s general double points of $\mathbf{P}^n$. Let S be the union of s general points of H. Let $E \subset H$ be the union of s general double points of H (not double points of $\mathbf{P}^n$, i.e. each of them has length n). To prove $h^1(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$) it is sufficient to prove $h^1(H, \mathcal{I}_{(Z\cap H)\cup S}(d)) = h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)\cup E}}(d-1)) = 0$ (resp. $h^0(H, \mathcal{I}_{(Z\cap H)\cup S}(d)) = h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)\cup E}}(d-1)) = 0$).*

Proof of Lemma 2. The case $d = 3$ follows from Proposition 1. Hence we will always assume $d \geq 4$. We divide the proof into three parts. In part (ii) we will describe the general inductive procedure to prove $A_{n,d}$ (or $A'_{n,d}$). We will use induction on n, but not on d. To check $A_{n,d}$ or $A'_{n,d}$ for some $n \geq 3$ and some integer $d \geq 4$ we will assume $A_{n-1,d}$, but only very weak forms of $A'_{n-1,d}$ and $A_{n-1,d-1}$. Instead, we will use the postulation with respect to $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d-1))$ of a certain zero-dimensional scheme, essentially a not quite general disjoint union of double points, simple points and length n zero-dimensional schemes which are double points of a fixed hyperplane $H \subset \mathbf{P}^n$. In part (i) we will use induction on d and the known cohomological properties of all subschemes of $\mathbf{P}^1$. In the inductive step from $A_{2,d-1}$ to $A_{2,d}$ and $A'_{2,d}$, $d \geq 5$, when d is even we will use a trick which will be heavily used in the general strategy of part (ii). To carry over the general strategy of part (ii) we will need some numerical assumptions, almost always satisfied (see Lemma 5 and Lemma 6); for the few remaining cases we will have to modify the constraction. In part (iii) we will check $A_{n,4}$ and $A'_{n,4}$; for very low n we will need direct numerical checkings.

(i) Here we will prove $A_{2,d}$ for every integer $d \geq 4$. Fix a line $D \in \mathbf{P}^2$. We have $x_{2,4} = 3$ and $y_{2,4} = 0$. Hence $A'_{2,4} = A_{2,4}$. Take $P, P_1 \in D$ with $P_1 \neq P$, and two general points $P_2, P_3 \in \mathbf{P}^2$ and set $Z := 3P \cup 2P_1 \cup 2P_2 \cup 2P_3$. Hence $\text{length}(Z \cap D) = 5$ and $W := \text{Res}_D(Z) = 2P \cup P \cup 2P_2 \cup 2P_3$. By Horace Lemma 3 we have $h^i(\mathbf{P}^2, \mathcal{I}_Z(4)) = h^i(\mathbf{P}^2, \mathcal{I}_W(3))$, $i = 0, 1$. Since P, P_2 and P_3 are not collinear, it is easy to check that the only degree 3 plane curve singular at P, P_2 and P_3 is the reducible curve union of the lines joining two of these points. Hence $h^i(\mathbf{P}^2, \mathcal{I}_W(3)) = 0$, $i = 0, 1$, proving $A_{2,4}$. Now assume $d \geq 5$ and that all assertions $A_{2,t}$ and $A'_{n,t}$, $4 \leq t \leq d-1$, are true. First assume d odd. Take $P_1, \ldots, P_{(d+1)/2} \in D$ and take as Z a general union of $2P_i$, $1 \leq i \leq (d+1)/2$, one triple point and $x_{2,d} - (d+1)/2$ further double points. Hence $\text{length}(Z \cap D) = d+1$ and $W := \text{Res}_D(Z)$ is the union of $P_1, \ldots, P_{(d+1)/2}$, a general triple point and $x_{2,d}$ general double points. Set $A := W \backslash \{P_1, \ldots, P_{(d+1)/2}\}$. To check $A_{2,d}$ it is sufficient to prove $h^1(\mathbf{P}^2, \mathcal{I}_W(d-1)) = 0$. Since $(d+1)/2 \leq d$ and the points $P_1, \ldots, P_{(d+1)/2}$ may be taken general in D, to prove $h^1(\mathbf{P}^2, \mathcal{I}_W(d-1)) = 0$ it is sufficient to prove $h^1(\mathbf{P}^2, \mathcal{I}_A(d-1)) = h^0(\mathbf{P}^2, \mathcal{I}_A(d-2)) = 0$ (easy or see [7], Lemma 3). Both vanishing are true by the inductive assumption and (for $d = 5$) the case $n = 2$ of Proposition 1. The proof of $A'_{2,d}$ is similar. Now assume d odd. Modify the first step of the proof using Lemma 4 as in [6] (called there the double residue trick) or [7], Lemma 4.

(ii) Fix $x_{n-1,d} + y_{n-1,d}$ general points P_i, $1 \leq i \leq x_{n-1,d}$, Q_j, $1 \leq j \leq y_{n-1,d}$, of H and let $W \subset H$ be the closed subscheme of H union of the points Q_j, $1 \leq j \leq y_{n-1,d}$, the double points of H with P_i as suports, $1 \leq i \leq x_{n-1,d}$, and the triple point of H supported by P. Thus $\text{length}(W) = y_{n-1,d} + nx_{n-1,d} + (n+1)n/2$. By (3) for the pair $(n-1, d)$ we obtain $\text{length}(W) = \binom{n+d-1}{n-1}$. Set $A := W \backslash \{Q_1, \ldots, Q_{y_{n-1,d}}\}$. By $A_{n-1,d}$ and Remark 1 we have $h^1(H, \mathcal{I}_A(d)) = 0$, i.e. $h^0(H, \mathcal{I}_A(d)) = y_{n-1,d}$. By the generality of the set $\{Q_1, \ldots, Q_{y_{n-1,d}}\} \subset H$ we obtain $h^0(H, \mathcal{I}_W(d)) = h^1(H, \mathcal{I}_W(d)) = 0$. Let B be the closed subscheme of $\mathbf{P}^n$ union of $3P$ and the double points $2P_i$, $1 \leq i \leq x_{n-1,d}$, (double points in $\mathbf{P}^n$, not in H, i.e. each of them has length $n+1$) and $x_{n,d} - x_{n-1,d} - y_{n-1,d}$ general double points of $\mathbf{P}^n$; here we use the inequality $x_{n,d} - x_{n-1,d} - y_{n-1,d} \geq 0$ proved in Lemma 5 below). Hence $\text{length}(B) = (n+2)(n+1)/2 + (n+1)(x_{n,d} - y_{n-1,d})$. Let E be the union of the double points of H (not of $\mathbf{P}^n$) supported by the points Q_j, $1 \leq j \leq y_{n-1,d}$. Hence $\text{length}(E) = ny_{n-1,d}$. Let Z be the the union of B and $y_{n-1,d}$ general double points of $\mathbf{P}^n$. By semicontinuity to prove $A_{n,d}$ it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$. Notice that $A = B \cap H$ (scheme-theoretically). By Lemma 4 and the vanishing of $h^1(H, \mathcal{I}_W(d))$ to prove $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{B \cup E}(d-1)) = 0$. We have $(B \cup E) \cap H = (B \cap H) \cup E$, i.e.

$(B \cup E) \cap H$ is the union of a double point of H supported by P, $y_{n-1,d}$ further general double points of H and $x_{n-1,d}$ general points of H. It is easy to check that $n(y_{n-1,d}+1)+x_{n-1,d} \le \binom{n+d-1}{n-1}$ (use (2) for the pair $(n-1,d)$). Hence to prove $h^1(H, \mathcal{I}_{(B\cup E)\cap H}(d-1)) = 0$ it is sufficient to prove $h^1(H, \mathcal{I}_M(d-1)) = 0$, where M is a general union of $y_{n-1,d}+1$ double points; this is easy and a very particular case of Alexander-Hirschowitz result ([2], [3], [4], [7]). Notice that $\mathrm{Res}_H(B\cup E) = \mathrm{Res}_H(B)$. Thus $\mathrm{Res}_H(B\cup E)$ is the union of $x_{n,d}-x_{n-1,d}-y_{n-1,d}$ general double points of $\mathbf{P}^n$. Hence $h^1(\mathbf{P}^n, \mathcal{I}_B(d-1)) = 0$ by a very weak form of $A_{n,d-1}$ true even if $d-1 = 3$ (see part (iii)). Since length$(E) = ny_{n-1,d}$ and $h^1(H, \mathcal{I}_{(B\cup E)\cap H}(d-1)) = 0$, to obtain $h^1(\mathbf{P}^n, \mathcal{I}_{B\cup E}(d-1)) = 0$, it is sufficient to use that $h^1(\mathbf{P}^n, \mathcal{I}_B(d-1)) = 0$ and prove $h^0(\mathbf{P}^n, \mathcal{I}_B(d-1)) \ge ny_{n-1,d}$ and $h^0(\mathbf{P}^n, \mathcal{I}_B(d-2)) \le h^0(\mathbf{P}^n, \mathcal{I}_B(d-1)) - ny_{n-1,d}$. The first inequality is obvious for degree reasons, while the second one is a very weak form of $A'_{n,d-2}$ true even for $d = 5$; for $d = 4$, see part (iii). The inductive proof of $A'_{n,d}$ is similar and omitted; we only remark that even for $A'_{n,d}$ we use $A_{n-1,d}$, not $A'_{n-1,d}$.

(iii) Here we will prove $A_{n,4}$ and $A'_{n,4}$ for all $n \ge 3$. From (3) we obtain $x_{3,4} = 6$, $y_{3,4} = 1$, $x'_{3,4} = 7$, $y'_{3,4} = 3$, $x_{4,4} = x'_{4,4} = 11$, $y_{4,4} = y'_{4,4} = 0$, $x_{5,4} = 17$, $y_{5,4} = 4$, $x'_{5,4} = 18$, $y'_{5,4} = 3$, $x_{6,4} = 24$, $y_{6,4} = 4$, $x'_{6,4} = 18$, $y'_{6,4} = 3$, $x_{7,4} = x'_{7,4} = 23$ and $y_{7,4} = y'_{7,4} = 0$. Hence $A'_{4,4} = A_{4,4}$ and $A'_{7,4} = A_{7,4}$. Fix a hyperplane H of $\mathbf{P}^n$ and $P \in H$. We will always take $3P$ as the triple point for $A_{n,4}$ and $A'_{n,4}$. First assume $n = 3$. We also take among the double points for $A_{3,4}$ (resp. $A'_{3,4}$) three double points supported by three general points of H; the remaining three (resp. four) double points will be supported by general points of $\mathbf{P}^3$. Call Z the union of all these schemes and Z' (resp. Z'') the subscheme of Z formed by the connected components whose reduction is (resp. is not) a point of H. Hence $\mathrm{Res}_H(Z) = \mathrm{Res}_H(Z') \cup Z''$ and $Z \cap H = Z' \cap H$. By $A_{2,4}$ we have $h^i(H, \mathcal{I}_{Z'\cap H}(4)) = 0$, $i = 0, 1$. Hence by Horace Lemma 3 to prove $A_{3,4}$ (resp. $A'_{3,4}$) it is sufficient to prove $h^1(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(Z)}(3)) = 0$ (resp. $h^0(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(Z)}(3)) = 0$); remember that the meaning of Z is different for $A_{3,4}$ and $A'_{3,4}$. For instance, for $A_{3,4}$ the scheme $\mathrm{Res}_H(Z)$ is the union of the double point $2P$, 4 general double points of $\mathbf{P}^3$ and 2 general points of H. To obtain $h^1(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(Z)}(3)) = 0$ just use a very particular case of the maximal rank for almost all general unions of double points in a projective space and [7], Lemma 3. Then we prove $A_{n,4}$ and $A'_{n,4}$ by induction on n taking $3P$ and, among the double points, $x_{n-1,4}$ with support on H. If $y_{n-1,4} = 0$ we are done as in the case $n = 3$. If $y_{n-1,4} > 0$, then we also use $y_{n-1,4}$ times the double residue trick (Lemma 4) with respect to general points of H. Hence for $\mathcal{O}_H(3)$ we must control the union of a double point supported by P, $x_{n-1,4}$ general points and $y_{n-1,4}$ general double points. The $y_{n-1,4}+1$ double points have good cohomology for $\mathcal{O}_H(3)$). To see that adding $x_{n-1,4}$ general points gives no trouble it is sufficient to have $n(y_{n-1,4}+1)+x_{n-1,4} \le h^0(H,\mathcal{O}_H(3)) = \binom{n+2}{3}$ and this is alway satified by the explicit values of $y_{n-1,4}$ even for low n. □

Lemma 5. *We have $x_{n,d} \ge x_{n-1,d} + y_{n-1,d}$ for all $n \ge 3$ and $d \ge 4$.*

Proof. Subtracting the equation in (2) for the pair $(n-1,d)$ from the same equation for the pair (n,d) we obtain:

$$n(x_{n,d}-x_{n-1,d}) + x_{n,d} + n + 1 + y_{n,d} - y_{n-1,d} = \binom{n+d-1}{n} \tag{5}$$

and hence it is sufficient to check the inequality

$$(n-1)y_{n-1,d} + x_{n,d} + y_{n,d} \le \binom{n+d-1}{n}. \tag{6}$$

Since $x_{n,d} \le (\binom{n+d}{n} - (n+2)(n+1)/2)/(n+1)$ (use (2)), $y_{n-1,d} \le n-1$ and $y_{n,d} \le n$, we easily conclude, at least if $n \ge 4$ and $d \ge 5$. For low n and d compute explicitly the values of the integers $x_{n,d}$, $x_{n-1,d}$ and $y_{n-1,d}$. □

Lemma 6. *We have*

$$x_{n-1,d} + n + ny_{n-1,d} \le \binom{n+d-2}{n-1}, \tag{7}$$

for all $n \ge 3$ and $d \ge 4$

Proof. By the equation in (3) for the pair $(n-1,d)$ we obtain that it is sufficient to use the trivial inequality $x_{n-1,d} \ge n-1$. □

Proof of Theorem 2. Apply Lemma 2 and Remark 2. □

Proof of Theorem 1. Apply Theorem 2 and Lemma 1. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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