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ON THE PRINCIPLE OF UNIFORM BOUNDEDNESS IN A STRICTLY $\mathcal{N}$-LOCALLY CONVEX SPACES

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Abstract: In this paper we would like to establish the uniform boundedness principle for sequentially continuous linear operators in some class of locally convex spaces (strictly $\mathcal{N}$-locally convex spaces).

If it is possible to prove the uniform boundedness principle in such spaces, then our result will generalize the uniform boundedness principle for continuous linear operators in Banach spaces, since the class of strictly $\mathcal{N}$-locally convex spaces contains the class of Banach spaces.

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1. Introduction

Let (X, λ) and (Y, μ) be locally convex spaces. An operator $T : X \to Y$ is said to be sequentially continuous if $\{x_n\}$ is a sequence in X such that $x_n \to x$ then $Tx_n \to Tx$; T is said to be bounded if T sends bounded sets into bounded sets.

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Clearly, continuous operators are sequentially continuous, and sequentially continuous operators are bounded but in general, converse implications fail. Let X'_λ, X^s_λ and X^b_λ denote the families of continuous linear functionals, sequentially continuous linear functionals and bounded linear functionals on (X, λ), respectively. In general, the inclusions $X'_\lambda \subset X^s_\lambda \subset X^b_\lambda$ are strict.

Let $\mathcal{B}(X_\lambda)$, $\mathcal{C}(X_\lambda)$ and $\mathcal{C}_0(X_\lambda)$ denote the families of bounded set in (X, λ), conditionally sequentially compact set in (X, λ) and convergent sequence in (X, λ) to 0, respectively.

By $\zeta_\sigma(\lambda)$ we denote the topology on X^b_λ generated by the family of seminorms

$$P_C(T) = \sup_{x \in C} \mid Tx \mid, \quad C \in \sigma.$$

Let $\beta(\lambda, \sigma)$ denote the topology of uniform convergence on $(X^s_\lambda, \zeta_\sigma(\lambda))$-bounded subsets of X^s_λ.

A locally convex space (X, τ) is said to be strictly $\mathcal{N}$-locally convex space if there exists a complete norm $\|.\|_X$ on X, a family σ of subsets of X and a locally convex topology λ on X such that

$$\mathcal{C}(X_\lambda) \subset \sigma \subset \mathcal{B}(X) \equiv \mathcal{B}(X, \|.\|_X) \subset \mathcal{B}(X_\lambda) \text{ and } \tau = \beta(\lambda, \sigma).$$

We can give many examples of such spaces. For example, if we take any Banach space E, then $(E, \|.\|_E) = (E, \beta(\|.\|_E, \mathcal{B}(E)))$ is a strictly $\mathcal{N}$-locally convex space.

Assume now that $(X, \beta(\lambda, \sigma))$ and $(Y, \beta(\mu, \sigma_1))$ are two strictly $\mathcal{N}$-locally convex spaces. Denote by $\mathcal{B}_1(Y)$ the unit ball in $(Y, \|.\|_Y)$. Let $l(X_\lambda, Y_\mu)$ and $l(X, Y)$ denote the families of Sequentially continuous linear mappings from (X, λ) to (Y, μ) and continuous linear mappings from $(X, \|.\|_X)$ to $(Y, \|.\|_Y)$.

Denote by X' the space of continuous linear functionals on $(X, \|.\|_X)$. Put $\mathcal{F}_1 = \mathcal{B}(Y^s_\mu, \zeta_{\sigma_1}(\mu))$. The following proposition shows that if $(X, \beta(\lambda, \sigma))$ is a strictly $\mathcal{N}$-locally convex space, then $X^s_\lambda \subset X' \equiv (X, \|.\|_X)'$ and every $\zeta_\sigma(\lambda)$-bounded subset of X^s_λ is bounded in $(X', \|.\|_{X'})$.

Proposition 1. *$X^s_\lambda \subset X' \equiv (X, \|.\|_X)'$. Moreover, $\mathcal{B}(X^s_\lambda, \zeta_\sigma(\lambda)) \subset \mathcal{B}(X', \|.\|_{X'})$.*

Proof. Suppose that $A \in X^s_\lambda$, but $A \notin X'$. Then,

$$\forall n \in N, \quad \exists x_n \in \mathcal{B}_1(X) \quad |Ax_n| \geq n^2.$$

Consequently, $(n^{-1}x_n) \in \mathcal{C}_0(X_\lambda)$. It follows then that $(An^{-1}x_n) \in \mathcal{C}_0(R)$. Thus $\{An^{-1}x_n\} \in \mathcal{B}(R)$, contradicting the fact that $|Ax_n| \geq n^2$.

Let us prove now that $\mathcal{B}(X^s_\lambda, \zeta_\sigma(\lambda)) \subset \mathcal{B}(X', \|.\|_{X'})$. Suppose the contrary. Then $\exists W \in \mathcal{B}(X^s_\lambda, \zeta_\sigma(\lambda))$ such that $W \notin \mathcal{B}(X', \|.\|_{X'})$. Consequently, $\forall n \in N$ $\exists A_n \in W$ $\exists x_n \in \mathcal{B}_1(X)$ such that $|A_n x_n| > n^2$. In this case $n^{-1}x_n \in \mathcal{C}_0(X) \subset \mathcal{C}_0(X_\lambda)$. Thus $\{n^{-1}x_n\} \in \mathcal{C}(X_\lambda) \subset \sigma$. Since W is bounded in $(X^s_\lambda, \zeta_\sigma(\lambda))$, it follows then that $\{A_n n^{-1} x_n\}$ is bounded, contradicting the fact that $|A_n x_n| > n^2$. Thus, we achieve the proof. □

It follows immediately from Proposition 1 that if $(Y, \beta(\mu, \sigma_1))$ is a strictly $\mathcal{N}$-locally convex space, then

$$\forall A \in \mathcal{F}_1, \quad \forall y \in Y \quad \mathcal{P}_A(y) \equiv \sup_{f \in A} \mid f(y) \mid < +\infty.$$

Thus, the topology $\beta(\mu, \sigma_1)$ is generated by the family $\mathcal{P}_A$, $A \in \mathcal{F}_1$ of semi-norms on Y.

It follows also from the definition of strictly $\mathcal{N}$-locally convex spaces and Proposition 1 that the topologies μ and $\beta(\mu, \sigma_1)$ are coarser than the norm $\|.\|_Y$.

2. Uniform Boundedness Principle in a Strictly $\mathcal{N}$ Locally Convex Spaces

Let $(X, \beta(\lambda, \sigma))$, $(Y, \beta(\mu, \sigma_1))$ two strictly $\mathcal{N}$ locally convex spaces. Let $T_n : X \to Y$ a sequence of linear operators. $\{T_n\}$ is said to be (σ_1, μ)-bounded if

$$\forall A \in \mathcal{F}_1, \quad \exists C_1 > 0, \quad \forall x \in \mathcal{B}_1(X), \quad \forall f \in A, \quad \forall n \in N \qquad \mid f(T_n x) \mid \leq C_1.$$

Assume for the moment that X, Y are two Banach spaces and assume also that there exists a sequence $T_n : X \to Y$ of continuous linear operators such that for some $x \in X$, the sequence $T_n x$ is not bounded in $(Y, \|.\|_Y)$. Then, we cannot apply the principle of uniform boundedness in Banach spaces. But, as we have seen before, if there exists a locally convex topology μ coarser than the norm $\|.\|_Y$ and a family σ_1 of subsets of Y such that

$$\mathcal{C}(Y_\mu) \subset \sigma_1 \subset \mathcal{B}(Y, \|.\|_Y) \subset \mathcal{B}(Y_\mu),$$

then μ and $\beta(\mu, \sigma_1)$ are coarser than the norm $\|.\|_Y$. Consequently, the sequence $(T_n x)$ can be bounded in (Y, μ) or in $(Y, \beta(\mu, \sigma_1))$ even if it is not bounded in $(Y, \|.\|_Y)$. Thus, we can apply the principle of uniform boundedness in the strictly $\mathcal{N}$-locally convex space $(Y, \beta(\mu, \sigma_1))$. We prove under the assumptions that $T_n : X \to Y$ is a sequence of (λ, μ)- sequentially continuous linear operators and $(T_n x)$ is bounded in $(Y, \beta(\mu, \sigma_1))$ for each $x \in X$ that T_n is (σ_1, μ)-bounded. It is the principle of uniform boundedness in a strictly $\mathcal{N}$-locally convex spaces.

Theorem 2. *Let $(X, \beta(\lambda, \sigma))$, $(Y, \beta(\mu, \sigma_1))$ two strictly $\mathcal{N}$ locally convex spaces and let $T_n : X \to Y$ a sequence of (λ, μ)- sequentially continuous linear operators, $n \in N$ such that the sequence $\{T_n x\}$ is bounded in $(Y, \beta(\mu, \sigma_1))$ for each $x \in X$. Then T_n is (σ_1, μ)-bounded.*

Remark 3. Let $x \in X$. We remark first that if the sequence $(T_n x)$ is bounded in $(Y, \beta(\mu, \sigma_1))$, then

$$\forall A \in \mathcal{F}_1 \text{ the sequence } (\sup_{f \in A} \mid f(T_n x) \mid) \text{ is bounded in } R.$$

Let us prove now Theorem 2.

Proof. Let $A \in \mathcal{F}_1$. Thus, using Remark 3, we deduce that

$$\forall x \in X \text{ the sequence } (\sup_{f \in A} \mid f(T_n x) \mid) \text{ is bounded in } R.$$

For every $s \in N$ we set

$$L_s = \{x \in X : \ \forall f \in A \ \forall n \in N \ \ \mid f(T_n x) \mid \leq s\}.$$

Thus, L_s is closed in $(X, \|.\|_X)$ and $\bigcup_{s \in N} L_s = X$. By Baire's Lemma, we deduce that there exists $s_0 \in N$ such that $\operatorname{int} L_{s_0} \neq \emptyset$. Let $x_0 \in X$ and $r > 0$ such that $\mathcal{B}(x_0, r) \subset L_{s_0}$. Therefore,

$$\forall n \in N, \ \ \forall f \in A, \ \ \forall z \in \mathcal{B}_1(X) \qquad r \mid f(T_n(z)) \mid \leq s_0 + \mid f(T_n x_0) \mid \leq M_1$$

for some $M_1 > 0$. Thus, we achieve the proof. □

In particular, if X, Y are two Banach spaces, setting $\lambda = \|.\|_X$, $\sigma = \mathcal{B}(X)$, $\mu = \|.\|_Y$, $\sigma_1 = \mathcal{B}(Y)$, we obtain the principle of uniform boundedness in Banach spaces. In this case $\beta(\mu, \sigma_1) = \|.\|_Y$, $\beta(\lambda, \sigma) = \|.\|_X$.

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