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SYMBOLIC DYNAMICS IN CHAOTIC WAVE VIBRATION

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Abstract: We consider a class of nonlinear boundary value problems for partial differential equations, whose solutions are, basically, characterized by the iteration of a nonlinear function. We apply methods of symbolic dynamics of discrete dynamical systems of the interval in order to compute the topological entropy associated to chaotic wave solutions. Then, from the variation of the entropy with the physical parameters we identify phase transitions. We also interpret the phase transition point in terms of the symbolic dynamics. Using these methods we formalize self-similar phenomena and study some of their properties.

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1. Introduction

In recent years, some examples of partial differential equations with nonlinear boundary conditions have been presented as a contribution to understand such basic notions as the concept of chaotic behavior for these kind of dynamical systems. The complexity of a solution is something one can realize just by looking at a phase space orbit, but it is not that easy to distinguish chaotic behavior among solutions corresponding to different values of the system parameters. Nevertheless this is something that is already been done in the theory of discrete dynamical systems. Thus, it is straightforward that the study of a nonlinear boundary value problem whose solutions are essentially determined by the iteration of a map can benefit from the mathematical chaos theory of discrete dynamical systems.

In this paper we consider the boundary value problem proposed by G. Chen, S.-B. Hsu and J. Zhou, [4], but we try a completely different approach by identifying a parameter value subset for which a dimensional reduction can be done. Then, we apply some ideas from the iteration of bimodal maps of the interval, in particular to the study of its topological invariants.

This paper are organized as follows: in Section 2, we present the problem proposed by Chen et al. Next, in Section 3, we identify a subset of parameter values for which the solution of the two-dimensional boundary value problem is essentially characterized by the iteration of a one-dimensional map. The aim of Section 4 is to apply symbolic dynamical techniques to study these maps of the interval. Finally, in Section 5, we study how the topological entropy changes with the parameters.

2. Preliminaries

Following [4], let $w(x,t)$ satisfy the wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2}, \quad -1 < x < 0,\ 0 < x < 1,\ t > 0,$$

with a boundary condition at the left endpoint $x = -1$,

$$w(-1,t) = 0, \quad t > 0,$$

a nonlinear boundary condition at the right endpoint $x = 1$,

$$w_x(1,t) = \alpha\, w_t(1,t) - \beta\, w_t^3(1,t), \quad 0 < \alpha \le 1,\ \beta > 0,\ t > 0,$$

a pair of transmission conditions at $x = 0$,

$$\begin{aligned} w_t(0^+,t) - w_t(0^-,t) &= -\eta\, w_x(0^+,t), \quad \eta > 0,\ \eta \neq 2,\ t > 0, \\ w_x(0^-,t) &= w_x(0^+,t), \end{aligned}$$

and some initial conditions

$$\begin{aligned} w(x,0) &= w_0(x), \\ w_t(x,0) &= w_1(x), \end{aligned} \qquad -1 < x < 1.$$

Now, let us introduce

$$\begin{aligned} w^{(1)}(x,t) &= -w(-x,t), \\ w^{(2)}(x,t) &= w(x,t), \end{aligned} \qquad x \in [0,1],\ t \geq 0,$$

and consider the Riemann variables

$$\begin{aligned} u_i(x,t) &= \tfrac{1}{2}(w_x^{(i)}(x,t) + w_t^{(i)}(x,t)), \\ v_i(x,t) &= \tfrac{1}{2}(w_x^{(i)}(x,t) - w_t^{(i)}(x,t)), \end{aligned} \qquad i = 1,2.$$

Therefore, the original problem is diagonalized into the first order hyperbolic system

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1(x,t) \\ u_2(x,t) \\ v_1(x,t) \\ v_2\,(x,t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_1(x,t) \\ u_2(x,t) \\ v_1(x,t) \\ v_2(x,t) \end{bmatrix},$$

for $0 < x < 1$, and $t > 0$, with initial conditions

$$\begin{bmatrix} u_1(x,0) \\ u_2(x,0) \\ v_1(x,0) \\ v_2(x,0) \end{bmatrix} = \begin{bmatrix} u_{1,0}(x) \\ u_{2,0}(x) \\ v_{1,0}(x) \\ v_{2,0}(x) \end{bmatrix},$$

and boundary conditions

$$\begin{aligned} \begin{bmatrix} u_1(1,t) \\ u_2(1,t) \end{bmatrix} &= \mathcal{R}_{1,\alpha\beta}(\begin{bmatrix} v_1(1,t) \\ v_2(1,t) \end{bmatrix}) \equiv \begin{bmatrix} v_1(1,t) \\ F_{\alpha\beta}(v_2(1,t)) \end{bmatrix}, \\ \begin{bmatrix} v_1(0,t) \\ v_2(0,t) \end{bmatrix} &= \mathcal{R}_{0,\eta}(\begin{bmatrix} u_1(0,t) \\ u_2(0,t) \end{bmatrix}) \equiv \begin{bmatrix} \frac{\eta}{2-\eta} & \frac{2}{2-\eta} \\ \frac{2}{2-\eta} & \frac{\eta}{2-\eta} \end{bmatrix} \begin{bmatrix} u_1(0,t) \\ u_2(0,t) \end{bmatrix}, \end{aligned}$$

where, for each $v \in \mathbb{R}$, the function $u = F_{\alpha\beta}\,(v)$ is the unique real solution of the cubic equation

$$\beta(u-v)^3 + (1-\alpha)(u-v) + 2v = 0,$$

that is,

$$F_{\alpha\beta}(v) = v + \sqrt[3]{-\frac{v}{\beta} + \sqrt{\frac{(1-\alpha)^3}{27\beta^3} + \frac{v^2}{\beta^2}}} + \sqrt[3]{-\frac{v}{\beta} - \sqrt{\frac{(1-\alpha)^3}{27\beta^3} + \frac{v^2}{\beta^2}}},$$

see [2] for details. From now on, we will denote both maps as $\mathcal{R}_0$, $\mathcal{R}_1$, instead of $\mathcal{R}_{0,\eta}, \mathcal{R}_{1,\alpha\beta}$, respectively. Thus, following again [2], we have, for $x \in [0,1]$, and $t = 2k + \tau$, with $k = 0, 1, 2, \ldots$, and $\tau \in [0, 2)$,

$$\begin{bmatrix} u_1(x,t) \\ u_2(x,t) \end{bmatrix} = \begin{cases} (\mathcal{R}_1 \circ \mathcal{R}_0)^k \begin{bmatrix} u_{1,0}(x+\tau) \\ u_{2,0}(x+\tau) \end{bmatrix}, & \tau \le 1 - x, \\ \mathcal{R}_0^{-1} \circ (\mathcal{R}_0 \circ \mathcal{R}_1)^{k+1} \begin{bmatrix} v_{1,0}(2-x-\tau) \\ v_{2,0}(2-x-\tau) \end{bmatrix}, & 1 - x < \tau \le 2 - x, \\ (\mathcal{R}_1 \circ \mathcal{R}_0)^{k+1} \begin{bmatrix} u_{1,0}(x+\tau-2) \\ u_{2,0}(x+\tau-2) \end{bmatrix}, & 2 - x < \tau \le 2, \end{cases}$$

$$\begin{bmatrix} v_1(x,t) \\ v_2(x,t) \end{bmatrix} = \begin{cases} (\mathcal{R}_0 \circ \mathcal{R}_1)^k \begin{bmatrix} v_{1,0}(x-\tau) \\ v_{2,0}(x-\tau) \end{bmatrix}, & \tau \le x, \\ \mathcal{R}_0 \circ (\mathcal{R}_0 \circ \mathcal{R}_1)^k \begin{bmatrix} u_{1,0}(\tau-x) \\ u_{2,0}(\tau-x) \end{bmatrix}, & x < \tau \le 1 + x, \\ (\mathcal{R}_0 \circ \mathcal{R}_1)^{k+1} \begin{bmatrix} v_{1,0}(x-\tau+2) \\ v_{2,0}(x-\tau+2) \end{bmatrix}, & 1 + x < \tau \le 2. \end{cases}$$

It follows from the formulae above that $w(x,t)$, the solution of our initial-boundary value problem, is essentially characterized by the iteration of the maps $\mathcal{R}_0 \circ \mathcal{R}_1$ and $\mathcal{R}_1 \circ \mathcal{R}_0$, hence inheriting the properties of these maps. For instance, if we choose parameter values $\alpha = 0.6$, $\beta = 1$, and $\eta = 0.82$, for which $\mathcal{R}_0 \circ \mathcal{R}_1$ has the strange attractor shown in Figure 1, one can presume a complex behavior for $w(x,t)$.

The general situation of this BVP is extremely difficult to handle, hence the qualitative analysis of the character of $w(x,t)$. In [4], Chen et al studied some elementary properties of the map $\mathcal{R}_0 \circ \mathcal{R}_1$, i.e., the existence and stability of its fixed points, being evident the hardness of such approach. With this work we show that, for some subsets of parameter values, it is not only possible to

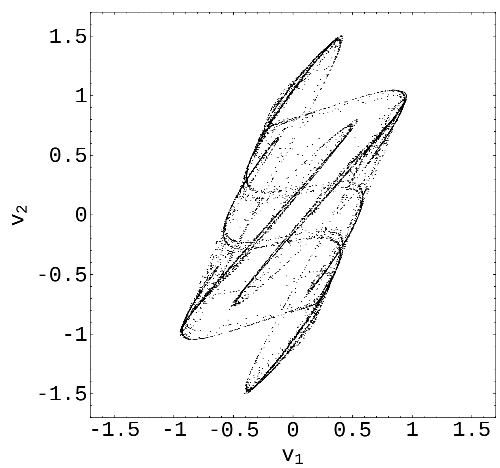

Figure 1: The strange attractor of $\mathcal{R}_0 \circ \mathcal{R}_1$ for $\alpha = 0.6$, $\beta = 1$, and $\eta = 0.82$

measure the complexity of $w(x,t)$ in terms of the associated topological entropy, but also to study how the entropy changes with the parameters. Despite the restricted subset of parameter values for which our study is valid, for small values of the parameter η, the injected energy is enough to excite periodic vibrations into chaos and we will be able to characterize these different chaotic solutions.

To finish this section, let us just point out that, since the maps $\mathcal{R}_0 \circ \mathcal{R}_1$, and $\mathcal{R}_1 \circ \mathcal{R}_0$, are dynamically equivalent, as proved in [4], our attention will be focused on the study of the map $\mathcal{R}_0 \circ \mathcal{R}_1$ alone.

3. Reduction to an One-Dimensional Problem

If we fix $\alpha = 0.6$, and $\beta = 1$, and consider a different value of η, say $\eta = 0.66$, we find that the attractor of $\mathcal{R}_0 \circ \mathcal{R}_1$ reveals a totally different picture. In fact, now the attractor is restricted to a set of segments, which means that only one of the two phase variables, v_1, and v_2, is really independent, i.e., that the problem can be reduced to a one-dimensional situation; see Figure 2. Without loss of generality, we choose v_1 as the independent variable for the 1-dimensional reduction. Thus, given an initial condition (v_1^0, v_2^0), let

$$(v_1^{k+1}, v_2^{k+1}) = (\mathcal{R}_0 \circ \mathcal{R}_1)^2 (v_1^k, v_2^k), \quad k = 1, 2, \dots .$$

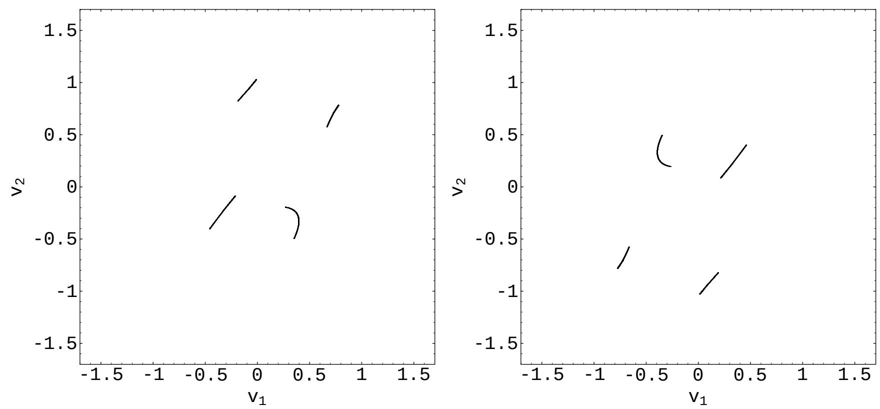

Figure 2: The strange attractors of $\mathcal{R}_0 \circ \mathcal{R}_1$, depending on the initial condition, for $\alpha = 0.6$, $\beta = 1$, and $\eta = 0.66$

For the above reduction to be valid, the graph of the points $(\upsilon_1^k, \upsilon_1^{k+1})$, after discarding some noninteresting transients, must reveal a functional dependence

$$\upsilon_1^{k+1} = \mathcal{H}(\upsilon_1^k).$$

Numerically, we observe that the points $(\upsilon_1^k, \upsilon_1^{k+1})$ suggest two symmetric cubic-like maps, depending on the chosen initial condition, that we denote by $\mathcal{H}_1$, and $\mathcal{H}_2$; see Figure 3 (this is the reason that it is more convenient to consider the map $(\mathcal{R}_0 \circ \mathcal{R}_1)^2$, instead of $\mathcal{R}_0 \circ \mathcal{R}_1$.) It is quite interesting to see the subsets D_1, and D_2, of the set of initial conditions $(\upsilon_1^0, \upsilon_2^0)$, that correspond to each of the maps. In Figure 4, we have a plot of D_1, and D_2, for $\alpha = 0.6$, $\beta = 1$, and $\eta = 0.66$.

But most important is the study of the subset of parameter values for which this reduction applies, i.e., the subset of parameter values for which one can find, asymptotically at least, that the solution $w(x,t)$ is mainly characterized by the iteration of a piecewise monotone map of the interval with two critical points, a bimodal map of the interval. Since β plays the simpler role of a scaling parameter, we can simplify the problem fixing its value, say $\beta = 1$. Thus, we found experimentally the subset $\mathcal{R}$ of the (α, η)-parameter space for which one has clearly a functional dependence between the variables (υ_1, υ_2), which was equivalent to $\mathcal{H}_1$, and $\mathcal{H}_2$ being bimodal maps of the interval. The result is depicted in Figure 5. As one can observe, $\mathcal{R}$ is very restrictive for the parameter η, only allowing small values for it. Since η is related with the energy injected into the system, the disturbance that causes chaotic solutions,

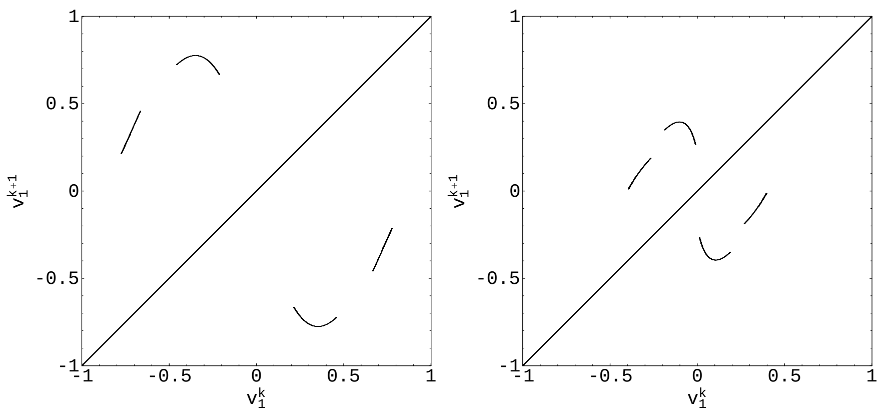

Figure 3: The maps $\mathcal{H}_1$ and $\mathcal{H}_2$, for $\alpha = 0.6$, $\beta = 1$, and $\eta = 0.66$

this means that our study is characterizing the very first chaotic regimes.

4. Symbolic Dynamics and Star Product

Since the pioneering work of Metropolis, Stein and Stein, [11], among others, it is understood that the complex evolution of a dynamical system can be most easily described in terms of a symbolic representation. Later, in 1977, it was Milnor and Thurston, [12], who, with its kneading theory, introduced the symbolic framework for the equivalence of piecewise monotone of maps of the interval. The interest of this symbolic classification was made clear by Guckenheimer, [7], who presented a classification theorem for unimodal maps of the interval based on its kneading sequences, showing with it how close the symbolic classification is from the topological one. Meanwhile, Derrida, Gervois, and Pomeau, [6], introduced a star product between unimodal symbolic sequences, see also [1], which provided a rigorous algebraic structure to describe the self-similarities first noticed by Feigenbaum. An analogous star product between bimodal symbolic sequences was pursued for years, been now understood to be quite settled, see [9], and [10]. In what follows we assume some knowledge of kneading theory of modal maps of the interval and of the bimodal star product. Thus, let us begin introducing some notation.

Consider a bimodal alphabet

$$\mathcal{A}_2 = \{L,\, A,\, M,\, B,\, R\},$$

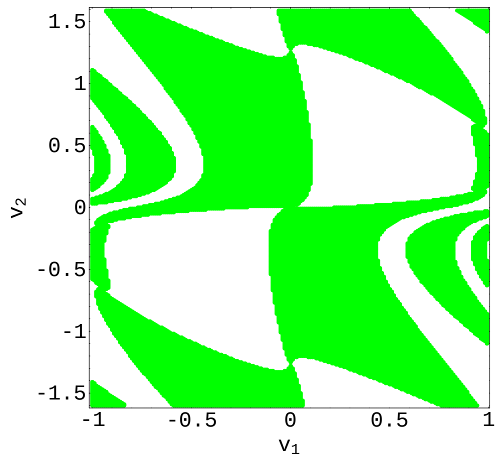

Figure 4: The subsets D_1, and D_2, for $\alpha = 0.6$, $\beta = 1$, and $\eta = 0.66$

with the symbols A, B corresponding to each of the critical points of the map. For the bimodal star product one needs also to consider the set of unimodal kneading sequences, hence, let

$$\mathcal{A}_1 = \{\ell, c, r\},$$

be a unimodal alphabet, with c the symbol for the unique critical point of the map. Finally, denote by $\mathcal{U}$ the set of unimodal kneading sequences, and by $\mathcal{B}_s$ the set of symmetric bimodal kneading data, i.e., the bimodal kneading data such that the symbolic orbit starting from the minimum is the conjugate of the one starting from the maximum of the map. With the following examples we present parameter values for which the bimodal kneading sequences are periodic (as usual, periodic bimodal kneading sequences will be denoted by finite sequences ended with a symbol A, or B).

Example 4.1. For $\alpha = 0.6$, and $\eta = 0.6617$, we have a map $\mathcal{H}_1$ with kneading sequences

$$\mathcal{K}(\mathcal{H}_1) = (RMRLRA,\ LMLRLB).$$

Example 4.2. For $\alpha = 0.6$, and $\eta = 0.6687$, we have a map $\mathcal{H}_1$ with kneading sequences

$$\mathcal{K}(\mathcal{H}_1) = (RMRLRLRA,\ LMLRLRLB).$$

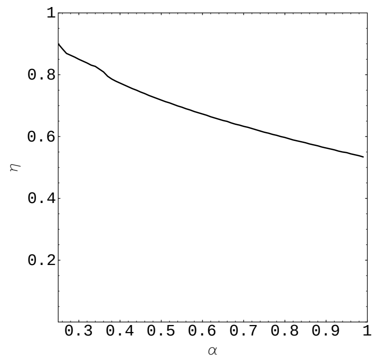

Figure 5: The subset $\mathcal{R}$ of the (α, η)-parameter space

Numerically we observed that the kneading sequences of both maps, $\mathcal{H}_1$, and $\mathcal{H}_2$, are equal, whenever the parameter values. This means that, although different maps, they share the properties taken from its kneading sequences. Therefore, our approach does not depend on the chosen initial condition $(\upsilon_1^0, \upsilon_2^0)$, and thus from now on we will only mention the map $\mathcal{H}_1$.

Definition 4.1. A pair (S, T) of sequences of symbols from the bimodal alphabet $\mathcal{A}_2$ is called $\mathcal{R}$-admissible if and only if

$$\exists\ (\alpha, \eta) \in \mathcal{R} \text{ such that } (S, T) = \mathcal{K}(\mathcal{H}_1),$$

with $\mathcal{H}_1$ the bimodal map corresponding to the parameter values (α, η).

From the symmetry of the maps $\mathcal{H}_1$ we can conclude that an $\mathcal{R}$-admissible bimodal kneading sequences must belong to the set $\mathcal{B}_s$.

Let us denote by $\partial\mathcal{R}^+$, and $\partial\mathcal{R}^-$, the upper, and lower boundary, respectively, of the parameter region $\mathcal{R}$. As one can see from Figure 5, $\partial\mathcal{R}^-$ corresponds to the parameter value $\eta = 0$.

Lemma 4.1. *For parameter values $(\alpha, \eta) \in \partial\mathcal{R}^-$, the map $\mathcal{H}_1$ has kneading data*

$$\mathcal{K}(\mathcal{H}_1) = \begin{cases} (M^\infty, M^\infty) & \text{if } 0 < \alpha < 1/2, \\ BA & \text{if } \alpha = 1/2, \\ ((RL)^\infty, (LR)^\infty) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Proof. If we consider values of η close to zero, then the map $\mathcal{R}_0$ comes close to a matrix with null diagonal elements, which means that the reduction described is approximately

$$\upsilon_1^{k+1} = \mathcal{H}(\upsilon_1^k) = F(\upsilon_1^k).$$

A close inspection shows that, for $0 < \alpha < 1$, the map F has an attractive period 2 orbit. Thus, from the symmetry of F we can conclude that, according to the position of the negative orbit point relative to its first critical point, the map F has kneading data either (M^∞, M^∞), if on the right, or $(BA)^\infty$, if exactly equal, or $((RL)^\infty, (LR)^\infty)$, if on the left. □

The study of the maps with parameter values from $\partial\mathcal{R}^+$ is much more difficult, but numerically we find that, for these (α, η), the map $\mathcal{H}_1$ has kneading data

$$\mathcal{K}(\mathcal{H}_1) = (RM(RL)^\infty, LM(LR)^\infty).$$

From the above and the definition of the symbolic bimodal star product, we can conclude the following lemma.

Lemma 4.2. *For any $\mathcal{R}$-admissible bimodal kneading data, $(S, \bar{S})$, there exists a unimodal kneading sequence, $s \in \mathcal{T}$, such that*

$$(S, \overline{S}) = (RA, LB) * (s, s),$$

where $*$ denotes the symbolic bimodal star product.

Proof. A $\mathcal{R}$-admissible bimodal kneading sequences $(S, \overline{S})$ lies necessary between $((RL)^\infty, (LR)^\infty)$, and $(RM(RL)^\infty, LM(LR)^\infty)$, and, from [9] and [10], we know that

$$((RL)^\infty, (LR)^\infty) = (RA, LB) * (\ell^\infty, \ell^\infty),$$
$$(RM(RL)^\infty, LM(LR)^\infty) = (RA, LB) * (r\ell^\infty, r\ell^\infty).$$

Therefore, any $\mathcal{R}$-admissible bimodal kneading sequences $(S, \overline{S})$ must be of the form $(RA, LB) * (s, s)$, for some unimodal kneading sequence s. □

Next, we will characterize the topological entropy of the family of maps $\mathcal{H}_1$, with parameter values from $\mathcal{R}$.

5. Entropy, Renormalization and Phase Transitions

By varying the parameter η increasingly from zero we are injecting more and more energy into the system. The next theorem characterizes the topological entropy of the family of maps $\mathcal{H}_1$.

Theorem 5.1. *For parameter values $(\alpha, \eta) \in \mathcal{R}$, the topological entropy of the map $\mathcal{H}_1$ varies between 0 and $\frac{1}{2}\ln(2)$.*

Proof. Consider an $\mathcal{R}$-admissible kneading sequences $(S, \overline{S})$. From the results given below, we conclude that the sequence S must satisfy the symbolic inequality

$$(RL)^{\infty} \preceq S \preceq RM(RL)^{\infty}. \tag{5.1}$$

But, again from [9] and [10], we can write these two bimodal sequences in terms of the bimodal star product as follows:

$$(RL)^{\infty} = RA * \ell^{\infty},$$
$$RM(RL)^{\infty} = RA * r\ell^{\infty},$$

and then, from the properties of the bimodal star product, we conclude that every bimodal kneading sequence satisfying (5.1) must be of the form $RA * s$, with s an unimodal kneading sequence such that $\ell^{\infty} \preceq s \preceq r\ell^{\infty}$, i.e., with s some unimodal kneading sequence. Therefore, there exists an unimodal kneading sequence s such that

$$S = RA * s. \tag{5.2}$$

Again from [9] and [10], we have that the topological entropy of a bimodal map $\mathcal{H}_1$ with kneading sequences (5.2) is given by

$$h_t(\mathcal{H}_1) = \frac{1}{2} h_t(f_s),$$

with f_s the unimodal map with kneading sequence s. The proof is complete, observing that the topological entropy of any unimodal map is always between zero and ln(2). □

With a numerical study (fitting of data), we are led to the following conjectures which show phase transitions.

Numerical Result 5.1. *i) As we can observe, by the Figure 6, the entropy increases with the parameter* η*. In fact, from our results we can conclude that the topological entropy of the map* $\mathcal{H}_1$ *is a three piecewise linear map in the parameter* η *with parameterization* $h_t(\eta) = c_1\eta + b_1(\alpha)$*, where the three different values of* c_1 *are* 0, $8.8\ldots$, $37.3\ldots$. *We conjecture a phase transition in this phenomena determined by the discontinuity in the derivative of the entropy as a function of* η *which is showed in the graph with three different slopes of the entropy as a function of* η.

ii) As we can observe, by the Figure 7, the entropy increases with the parameter α*. The topological entropy of the map* $\mathcal{H}_1$ *is a three piecewise linear map in the parameter* α *with parameterization* $h_t(\alpha) = c_2\alpha + b_2(\eta)$*, where the three different values of* c_2 *are* 0, $4.04\ldots$, $16.7\ldots$. *We conjecture a phase transition in this phenomena determined by the discontinuity in the derivative*

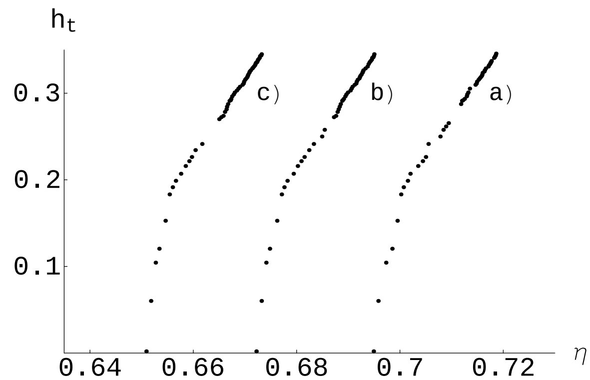

Figure 6: Graph of the topological entropy of the map $\mathcal{H}_1$, with $\beta = 1$, $\eta \in [0.6424, 0.7186]$, a) $\alpha = 0.5$, b) $\alpha = 0.55$, c) $\alpha = 0.6$

of the entropy as a function of α *which is showed in the graph with three different slopes of the entropy as a function of* α.

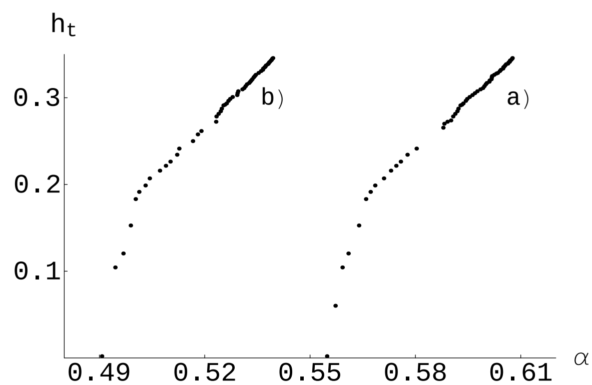

Figure 7: Graph of the topological entropy of the map $\mathcal{H}_1$, with $\beta = 1$, $\alpha \in [0.4968, 0.6076]$, a) $\eta = 0.67$, b) $\eta = 0.7$

iii) As we can observe, by the Figure 8, if we fix the topological entropy in a constant value, the parameter η *decreases with the parameter* α. *The parameter* η *is a linear map in the parameter* α *with parameterization* $\eta(\alpha) = c_3\alpha + b_3(h_t)$, *where* $c_3 = -0.45\ldots$.

Numerical Result 5.2. *The second phase transition occurs where the dynamics realizes the kneading pair* $(RA, LB) * (rlr^\infty, rlr^\infty) = (RMRL(RM)^\infty, LMLR(LM)^\infty)$ *with topological entropy equal to*

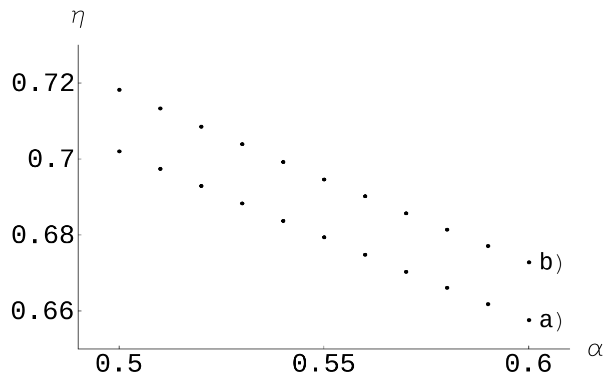

Figure 8: Graph of the map $\eta(\alpha)$, with $\beta = 1$, $\alpha \in [0.5, 0.6]$, a) $h_t(\mathcal{H}_1) = 0.207\ldots$, b) $h_t(\mathcal{H}_1) = 0.341\ldots$

$(\log 2)/4$.

6. Conclusion and Discussion

Although the class of boundary value problems which solutions are reduced to the iteration of some functions are rather simple, it is known, [3], [4] and [18], that its solutions exhibit very complex behavior. Here, we illustrated how one can use symbolic dynamical techniques and measure the chaoticity of its solutions. The challenge now is to systematically use this kind of technique to study PDE solutions in order to completely characterize their topological complexity.

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