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HOLOMORPHIC VECTOR BUNDLES AND PROPER HOLOMORPHIC MAPS

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Abstract: Here we explore the following definition. Let $f : X \to Z$ be a proper holomorphic map between reduced complex spaces such that $f_*(\mathcal{O}_X) = \mathcal{O}_Z$ and E a holomorphic vector bundle on X. We will say that E is *new* for f if there is no holomorphic vector bundle F on Z such that $f^*(F) \cong \mathcal{O}_Z$. We give a few cases in which there are holomorphic vector bundles on X with low rank and new for f. If f is projective, we even show the existence of topologically trivial vector bundles which are new for f.

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1. Bundles and Proper Maps

Here we explore the following definition.

Definition 1. Let $f : X \to Z$ be a proper holomorphic map between reduced complex spaces such that $f_*(\mathcal{O}_X) = \mathcal{O}_Z$ and E a holomorphic vector bundle on X. We will say that E is *new* with respect to f or for f if there is no holomorphic vector bundle F on Z such that $f^*(F) \cong \mathcal{O}_Z$.

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Remark 1. Take $f : X \to Z$ and E as in Definition 1. It is easy to check that f is surjective. The condition "$f_*(\mathcal{O}_X) = \mathcal{O}_Z$" is the right scheme-theoretical condition meaning that all fibers of f are connected. Since $f_*(\mathcal{O}_X) = \mathcal{O}_Z$ and any holomorphic vector bundle on Z is locally trivial, E is not new for f if and only if for every $Q \in Z$ there is an open neighborhood U of Q in Z such that $E|f^{-1}(U)$ is trivial. The condition "$f_*(\mathcal{O}_X) = \mathcal{O}_Z$" implies that Z is normal if X is normal (use the theory of the analytic spectrum for coherent analytic sheaves of algebras).

In the set-up of Definition 1, set $\mathrm{Exc}(f) := \{P \in X : f$ is not an isomorphism at $P\}$. Hence $\mathrm{Exc}(f) = X$ if and only if $\dim(Z) < \dim(X)$.

Let X be a holomorphically convex reduced complex space. Up to isomorphisms there is a unique pair (Z, f), where Z is a reduced Stein complex space, f is a proper map and $f_*(\mathcal{O}_X) = \mathcal{O}_Z$. The map $f : X \to Z$ is called the Remmert reduction of X and it is uniquely determined by the fact that for every $P \in X$ the set-theoretic fiber $f^{-1}(f(P))$ is the union of all connected compact analytic subsets of X containing P. The proofs of [1], Theorem 1.1 and Theorem 1.2, give verbatim respectively the following results.

Theorem 1. *Let X be a connected holomorphically convex manifold and $f : X \to Z$ its Remmert reduction. Assume* $\mathrm{Exc}(f) \neq \emptyset$ *(i.e. that X is not Stein) and* $\mathrm{Exc}(f) \neq X$ *(i.e.* $\dim(Z) = \dim(X)$*. If* $\mathrm{Exc}(f)$ *contains a hypersurface of X, then there is a holomorphic line bundle on X which is new for f. If* $\mathrm{Exc}(f)$ *does not contain a hypersurface of X, then TX is new for f.*

Theorem 2. *Let $f : X \to Z$ be a proper morphism between complex spaces such that $f_*(\mathcal{O}_X) = \mathcal{O}_Z$, X is smooth and conected and* $\dim(X) - 2 \leq \dim(Z) \leq \dim(X) - 1$*. Then for every open Stein subset U of Z either there is a holomorphic line bundle on $f^{-1}(U)$ which is new for $f|f^{-1}(U)$ or the tangent bundle $Tf^{-1}(U)$ is new for $f|f^{-1}(U)$.*

In the set-up of Theorem 2 if $Tf^{-1}(U)$ is new for $f|f^{-1}(U)$, then TX is new for f. In the case $\dim(Z) = \dim(X) - 1$ we may give the following more precise result.

Theorem 3. *Let $f : X \to Z$ be a proper morphism between complex spaces such that $f_*(\mathcal{O}_X) = \mathcal{O}_Z$, X is reduced and irreducible and* $\dim(Z) = \dim(X) - 1$*. Fix $Q \in Z_{reg}$ such that the scheme-theoretic fiber $f^{-1}(Q)$ is reduced, one-dimensional and contained in X_{reg}. Then there is a an open neighborhood U of Q in Z and a holomorphic rank two bundle on $f^{-1}(U)$ which is topologically trivial, but not holomorphically trivial. The same is true for rank one if there is an irreducible component T of $f^{-1}(Q)$ such that*

$p_a(T) > 0$, *but it is false if* $f^{-1}(Q) \cong \mathbf{P}^1$.

Proof. Let C be a reduced, connected and compact curve and R a holomorphic line bundle on C. R is topologically trivial if and only if $\deg(R|T) = 0$ for every irreducible component T of C. Since $f_*(\mathcal{O}_X) = \mathcal{O}_Z$, $f^{-1}(Q)$ is connected and hence we may apply the previous part. Fix an irreducible component T of $f^{-1}(Q)$ and fix any $P \in T \cap f^{-1}(Q)_{reg}$. Since X is smooth at P, Z is smooth at Q and the scheme-theoretic fiber $f^{-1}(Q)$ is smooth and of dimension $\dim(X) - \dim(Z)$ at P, f is smooth at P. Hence by the Inverse Function Theorem there is an open neighborhood of Q in Z and a morphism $g : U \to X$ such that $g(Q) = P$ and $f \circ g = \mathrm{Id}_U$. Set $E := \mathcal{O}_{f^{-1}(U)}(g(U)) \oplus \mathcal{O}_{f^{-1}(U)} - (g(U))$. E is topologically trivial, but it is new for $f|f^{-1}(U)$ because $E|f^{-1}(Q)$ is not holomorphically trivial. Now assume the existence of an irreducible component T of $f^{-1}(Q)$ such that $p_a(T) > 0$. There are P_1 and $P_2 \in T \cap f^{-1}(Q)_{reg}$ such that the the line bundle $\mathcal{O}_{f^{-1}(Q)}(P_1 - P_2)$ is not holomorphically trivial. By the Inverse Function Theorem there are $g_i : U \to X$, $i = 1, 2$, such that $g(Q) = P_i$ and $f \circ g_i = \mathrm{Id}_U$. Use the line bundle $\mathcal{O}_{f^{-1}(U)}(g_1(U) - g(U_2))$. Now assume $f^{-1}(Q) \cong \mathbf{P}^1$. By the rigidity of $\mathbf{P}^1$ there is an open polydisk U around Q and an isomorphism $f^{-1}(U) \cong U \times \mathbf{P}^1$ which preserve f. Every topologically trivial holomorphic line bundle on $U \times \mathbf{P}^1$ is holomorphically trivial, concluding the proof. □

When the morphism $f : X \to Z$ is projective in the sense of [3], Chapter IV, the existence of a holomorphic line bundle on X not coming from f (not even locally over Z) is guaranteed by the very definition of projective morphism. We may ask for a different problem, requiring also other conditions on the vector bundle E, e.g. that it is topologically trivial locally over Z. We have the following result.

Theorem 4. *Let* $f : X \to Z$ *be a projective morphism between reduced and irreducible complex spaces such that* $f_*(\mathcal{O}_X) = \mathcal{O}_Z$ *and* $r := \dim(X) - \dim(Z) > 0$. *Fix* $Q \in Z$ *such that the scheme-theoretic fiber* $f^1(Q)$ *is reduced and of dimension* r. *Then there is an open neighborhood* U *of* Q *in* Z *and a topologically trivial rank* $r+1$ *holomorphic vector bundle* E *on* $f^{-1}(U)$ *such that for every open neighborhood* V *of* Q *in* U *the holomorphic bundle* $E|f^{-1}(V)$ *does not come from* $f|f^{-1}(V)$.

Proof. Let $\mathbf{P}^r$ be the r-dimensional complex projective space. Twisting by $\mathcal{O}_{\mathbf{P}^r}(1)$ the dual of the Euler's sequence of the tangent bundle of $\mathbf{P}^r$ we obtain the following exact sequence

$$0 \to \Omega^1_{\mathbf{P}^r}(1) \to \mathcal{O}_{\mathbf{P}^r}^{\oplus(r+1)} \to \mathcal{O}_{\mathbf{P}^r}(1) \to 0\,. \qquad (1)$$

Set $A := \Omega^1_{\mathbf{P}^r}(1) \oplus \mathcal{O}_{\mathbf{P}^r}(1)$. Call

$$\epsilon \in \mathrm{Ext}^1(\mathcal{O}_{\mathbf{P}^r}(1), \Omega^1_{\mathbf{P}^r}(1)) \cong H^1(\mathbf{P}^r, \Omega^1_{\mathbf{P}^r}) \cong \mathbf{C}$$

the extension class inducing the exact sequence (1). For every scalar t let A_t be the middle term of the exact sequence induced by the extension class $t\epsilon$. By the very definition of ϵ we have $A_0 = A$ and $A_1 \cong \mathcal{O}_{\mathbf{P}^r}^{\oplus(r+1)}$. However, for $t \neq 0$ the bundles A_t and A_1 are isomorphic: use the endomorphism of A_1 induced by the multiplication by t on each fiber of A_1. Furthermore, $A_1 \cong \mathcal{O}_{\mathbf{P}^r}^{\oplus(r+1)}$. Since any exact sequence of holomorphic vector bundles splits as a sequence of topological vector bundles, A is topologically trivial. For any open neighborhood W of Q in Z let $\pi_W :: W \times \mathbf{P}^r \to \mathbf{P}^r$ and $q_W : W \times \mathbf{P}^r \to W$ be the projections. Set $A_W := \pi_W^*(A)$. Fix any morphism $g : Y \to \mathbf{P}^r$ and set $B := f^*(A)$. B is topologically trivial and it has a positive factor $g^*(\mathcal{O}_{\mathbf{P}^r}(1))$. For any morphism $u : f^{-1}(W) \to W \times \mathbf{P}^r$ such that $q_W \circ u = f|f^{-1}(W)$ the bundle $u^*(A_W)$ is topologically trivial, but its restriction to each fiber of $f|f^{-1}(W)$ has a positive rank one factor and hence it is new for $f|f^{-1}(W)$. Thus to conclude the proof it is sufficient to prove the existence of such a pair (W, u). By assumption there is a relatively f-ample line bundle L on X. For any relatively compact open neighborhood W of Q there is an integer m_0 such that for all integers $m \geq m_0$ we have $R^i h_*(L^{\otimes m}|f^{-1}(W)) = 0$ for all $i > 0$ and the natural map $\alpha : h^*(h_*(L^{\otimes m}|f^{-1}(W))) \to L|f^{-1}(W)$ is surjective and commutes with base change (e.g. restriction to any fiber), where $h := f|f^{-1}(W)$. Furthermore, $h_*(L^{\otimes m}|f^{-1}(W))$ is locally free. On $f^{-1}(Q)$ we obtain $r+1$ sections spanning $L^{\otimes m}|f^{-1}(Q)$ and hence a non-constant morphism $u' : f^{-1}(Q) \to \mathbf{P}^r$. Take a small neighborhood of Q on which $h_*(L^{\otimes m}|f^{-1}(W))$ and construct the map u on this neighborhood using u'. □

Remark 2. Use the notation of Theorem 4. If $r \leq 3$, then we may take E as above with $\mathrm{rank}(E) = 2$ by [2].

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References

[1] E. Ballico, Holomorphic vector bundles on certain holomorphically convex complex manifolds, *Bulletin U.M.I.*, To Appear.

[2] C. Banica, Topologisch triviale holomorphe Vectorbündel auf $\mathbf{P}^n(\mathbf{C})$, *J. Reine und Angew. Math.*, **344** (1983), 102-119.

[3] C. Bănica, O. Stănăsilă, *Algebraic Methods in the Global Theory of Complex Spaces*, John Wiley and Sons, London-New York (1976).