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APPLICATION OF A TSCHIRNHAUSIAN TRANSFORM TO QUARTIC EQUATIONS

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Abstract: Tschirnhausian transform of a quadratic polynomial type is applied to quartic equations to obtain all roots algebraically. The subsidiary equation for a parameter is given as a cubic equation explicitly in terms of coefficients of the said quartic equation, with an exception that the quartic becomes a compound quadratic in a parallel shift.

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1. Introduction

Historically Tschirnhausian transform is used to produce a Bring form (the canonical form) in quintic equations [1], [2] to obtain roots in terms of modular functions. Here, Tschirnhausian transform (different from a standard form) is applied to quartic equations to produce intermediately a compound quadratic equation.

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2. Analysis

2.1. General

Let a given quartic equation be expressed as

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0\,. \tag{1}$$

The quantity $A^3 - 4AB + 8\,C$ corresponding to the polynomial $x^4 + Ax^3 + Bx^2 + Cx + D$ is an invariant with respect to the shift (parallel translation) $x \to x+$ constant.

3. The Case of $A^3 - 4AB + 8C \neq 0$

Let a function $E(u, v)$ be defined by

$$E(u,v) = \begin{vmatrix} 1 & A & B & C & D & 0 \\ 0 & 1 & A & B & C & D \\ 1 & u & v & 0 & 0 & 0 \\ 0 & 1 & u & v & 0 & 0 \\ 0 & 0 & 1 & u & v & 0 \\ 0 & 0 & 0 & 1 & u & v \end{vmatrix}\,. \tag{2}$$

Consider a Tschirnhausian transform

$$-y = p + qx + x^2. \tag{3}$$

Eliminating x from equation (1) and equation (3) gives

$$E(q, p + y) = 0, \tag{4}$$

which can be expressed as a polynomial of y as follows:

$$\sum_{n=0}^{4} G_n y^n = 0, \tag{5}$$

where

$$G_4 = 1, \tag{6}$$

$$G_3 = 4p - Aq - 2B + A^2, \tag{7}$$

$$\begin{aligned} G_2 = {} & 6p^2 - 3Apq + Bq^2 \\ & +(-6B + 3A^2)p + (3C - AB)q + 2D - 2AC + B^2, \end{aligned} \tag{8}$$

$$\begin{aligned} G_1 = {} & 4p^3 - 3Ap^2q + 2Bpq^2 - Cq^3 \\ & +(3A^2 - 6B)p^2 + (6C - 2AB)pq + (-4D + AC)q^2 \\ & +(4D - 4AC + 2B^2)p + (3AD - BC)q \\ & -2BD + C^2, \end{aligned} \tag{9}$$

$$G_0 = E(q,p). \tag{10}$$

Let p and q satisfy simultaneously

$$G_3 = G_1 = 0, \tag{11}$$

that is,

$$p = \frac{1}{4}\left(Aq + 2B - A^2\right), \tag{12}$$

$$(A^3 - 4AB + 8C)F_1(A,B;q) + 8F_2(A,B,C,D;-q/2) = 0, \tag{13}$$

$$F_1(A,B;z) \equiv 2z^3 - 3Az^2 + (3A^2 - 4B)z - A(A^2 - 2B), \tag{14}$$

$$\begin{aligned} F_2(A,B,C,D;z) \equiv {} & (A^3 - 4AB + 8C)z^3 \\ & +(A^2B + 2AC - 4B^2 + 16D)z^2 \\ & +(A^2C + 8AD - 4BC)z \\ & +A^2D - C^2, \end{aligned} \tag{15}$$

where the function F_2 is just identical to the subsidiary one appearing in the reciprocal solution of equation (1), see [3]. Equation (13) is a cubic equation in q, the coefficient of the third power of which is $A^3 - 4AB + 8C(\neq 0)$. Thus q can be given as any one of the roots of equation (13). In case A, B, C, and D are all real, for equation (13) at least one real root exists, which may be used, but selection is not restricted to it. For a specified q, p is given by equation (12), and consequently corresponding four roots y_1, y_2, y_3, y_4 of equation (5) (a compound quadratic equation) are given by

$$\left\{ \begin{array}{c} y_1 \\ y_2 \end{array} \right\} = \sqrt{\frac{1}{2}\left(-G_2 \pm \sqrt{{G_2}^2 - 4G_0}\right)}, \tag{16}$$

$$\left\{ \begin{array}{c} y_3 \\ y_4 \end{array} \right\} = -\sqrt{\frac{1}{2}\left(-G_2 \pm \sqrt{{G_2}^2 - 4G_0}\right)}. \tag{17}$$

From each y_i $(i = 1, \ldots, 4)$ the roots of equation (1) are formed as follows: Let P_i and Q_i be defined as

$$P_i = -q^3 + 2(p + y_i)q + A(q^2 - p - y_i) - Bq + C, \tag{18}$$

$$Q_i = -(p + y_i)(q^2 - p - y_i) + A(p + y_i)q - B(p + y_i) + D, \tag{19}$$

where P_i, Q_i are introduced as coefficients of the residual of the quotient $x^4 + Ax^3 + Bx^2 + Cx + D$ divided by $x^2 + qx + p + y_i$. In case of $P_i \neq 0$, only one root is produced by

$$-Q_i/P_i,$$

whereas in case of $P_i = 0$, two roots are produced by

$$\frac{1}{2}\left\{-q \pm \sqrt{q^2 - 4\,(p + y_i)}\right\},$$

where at least one is multiply counted as an element of the set of roots, inclusively in case of a double-roots case.

3.1. In Case of $A^3 - 4AB + 8C = 0$

Translation $x = z - A/4$ leads to a compound quadratic equation in z:

$$z^4 + \left(-\frac{3}{8}A^2 + B\right) z^2 + \frac{5}{256}A^4 - \frac{A^2 B}{16} + D = 0. \tag{20}$$

4. Conclusion

Tschirnhausian transform is applied to quartic equations, to give roots algebraically through a subsidiary cubic equation via a compound quadratic equation.

References

[1] Arthur Cayley, On Tschirnhaus's transformation, *Phil. Trans. Roy. Soc. London*, **152** (1862), 561-578.

[2] Harold T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover (1962), 172-174.

[3] Y. Mochimaru, Reciprocal solution of a quartic equation, *International J. of Pure and Applied Mathematics*, **14** (2004), 209-212.

Appendix

(A) Function E

$$\begin{aligned} E(q,p) &= p^4 - Ap^3q + Bp^2q^2 - Cpq^3 + Dq^4 \\ &+(A^2-2B)p^3 + (3C-AB)p^2q \\ &+(AC-4D)pq^2 - ADq^3 \\ &+(B^2-2AC+2D)p^2 + (3AD-BC)pq + BDq^2 \\ &+(C^2-2BD)p - CDq + D^2. \end{aligned} \tag{21}$$