SEPARATION IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract: The basic concepts of the theory of intuitionistic fuzzy topological spaces have been defined by D. Çoker and co-workers. In a recent paper, we define two notions of Hausdorffness in the intuitionistic fuzzy sense, and obtain some new properties. In this paper we introduce normality and regularity in the intuitionistic fuzzy sense and obtain relations between these concepts and also with the fuzzy notions.

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1. Introduction

The introduction of “intuitionistic fuzzy sets” is due to K.T. Atanassov [1], and this theory has been developed by various authors [2-4]. In particular D. Çoker has defined the intuitionistic fuzzy topological spaces, and several authors have studied this category [5-11, 13, 15-17].

On separation in intuitionistic fuzzy topological spaces only there exists a paper with two kinds of Hausdorffness [17]. Then, in this new paper, we define regularity and normality for IFTS and obtain various properties between these concepts and also the relations with the fuzzy notions.

Firstly, we list some previous definitions.
Definition 1. (see [7]) Let \( X \) be a nonempty set and \( c \in X \) a fixed element in \( X \). If \( \alpha \in (0, 1) \) and \( \beta \in [0, 1) \) are two fixed real numbers such that \( \alpha + \beta \leq 1 \), then the IFS
\[
c(\alpha, \beta) = (x, c_{\alpha}, 1 - c_{1 - \beta})
\]
is called an intuitionistic fuzzy point (IFP for short) in \( X \).

If \( \beta \in [0, 1) \) is a fixed real number, then the IFS
\[
c(\beta) = (x, 0, 1 - c_{1 - \beta})
\]
is called a vanishing intuitionistic fuzzy point (VIFP for short) in \( X \).

Definition 2. (see [7]) (a) Let \( c(\alpha, \beta) \) be an IFP in \( X \) such that \( \alpha, \beta \in (0, 1) \) and \( A = (x, \mu_A, \gamma_A) \) be an IFS in \( X \). \( c(\alpha, \beta) \) is said to be properly contained in \( A \) (short) if \( \alpha < \mu_A(c) \) and \( \beta > \gamma_A(c) \).

(b) Let \( c(\beta) \) be an IFS in \( X \). \( c(\beta) \) is said to be properly contained in \( A \) (short) if \( \mu_A(c) = 0 \) and \( \beta > \gamma_A(c) \).

Definition 3. (see [7]) Let \( (X, \tau) \) be an IFTS on \( X \), and \( N \) be an IFS in \( X \). \( N \) is said to be an \( \varepsilon \)-neighborhood of an IFP \( c(\alpha, \beta) \) in \( X \) if there exists an IFOS \( G \) in \( X \) such that \( c(\alpha, \beta) \in G \subseteq N \). \( N \) is said to be an \( \varepsilon \)-neighborhood of a VIFP \( c(\beta) \) in \( X \) if \( \mu_N(c) = 0 \) and there exists IFOS \( G \) in \( X \) such that \( c(\beta) \in G \subseteq N \).

Definition 4. (see [17]) An IFTS \( (X, \tau) \) is called \( T_2 \) if for every IFPs or VIFPs \( p, q \) in \( X \) such that \( p \neq q \), there exist \( \varepsilon \)-neighborhoods \( M \) and \( N \) of \( p \) and \( q \) respectively such that \( M \cap N = 0 \) (clearly, for \( c(\alpha, \beta) \) we discard \( \alpha = 1, \beta = 0 \), and for \( c(\beta) \) the case \( \beta = 0 \)).

The next definitions are new concepts introduced in this paper.

Definition 5. An IFTS \( (X, \tau) \) will be called regular if for each IFP \( p \) and each IFCS such that \( p \cap C = 0 \), there exist IFOSs \( M \) and \( N \) such that \( p \subseteq M, C \subseteq N \) and \( M \cap N = 0 \).

Definition 6. An IFTS \( (X, \tau) \) will be called normal if for each IFCSs \( C_1 \) and \( C_2 \) such that \( C_1 \cap C_2 = 0 \), there exist IFOSs \( M_i \) and \( M_i \) such that \( C_i \subseteq M_i \) \( (i = 1, 2) \) and \( M_1 \cap M_2 = 0 \).

Lemma 1. Let \( A \) be an IFS in a IFTS \( (X, \tau) \). Then, \( A \) is IFOS if and only if for every IFP or VIFP \( p \in A \) there exists an \( \varepsilon \)-neighborhood \( M_p \) of \( p \) such that \( M_p \subseteq A \).

Proof. The first implication is clear. Conversely, for each \( p \in A \) there is an IFOS and \( \varepsilon \)-neighborhood \( M_p \) of \( p \) such that \( M_p \subseteq A \), then \( A = \bigcup_{p \in A} M_p \) because
if for each IFP or VIFP \( p \in A \) there is an IFOS \( M_p \) such that \( p \in M_p \subseteq A \), we have that

\[
A = \bigcup_{p \in A} \bigcup_{p \in A} M_p.
\]

**Proposition 1.** Let \((X, \tau)\) be a \(T_2\) IFTS. Then, each IFP \( p \) in \( X \) is an IFCS in \((X, \tau)\).

**Proof.** Let \( p = c(\alpha, \beta) \) such that \( \alpha \in (0,1], \beta \in [0,1) \) and \( \alpha + \beta \leq 1 \), then

\[
p = \langle x, c, 1 - c - \beta \rangle.
\]

We will apply Lemma 1 to prove that \( p \) is IFOS in \((X, \tau)\).

For every IFP \( q = d(\gamma, \delta) \) such that \( q \in p = \langle x, 1 - c - \beta, c \rangle \) we have that

\[
\begin{align*}
\gamma < (1 - c - \beta)(d), \\
\delta > c(\alpha)(d).
\end{align*}
\]

If \( q = p \), then \( d = c, \gamma = \alpha, \delta = \beta \), and

\[
\begin{align*}
\alpha < (1 - c - \beta)(c) = \beta, \\
\beta > c(\alpha)(c) = \alpha,
\end{align*}
\]

and this is contradictory.

Thus, \( q \neq p \). By the hypothesis there exist \( \varepsilon \)-neighborhoods \( M \) and \( N \) of \( p \) and \( q \) respectively such that \( M \cap N = 0 \), then

\[
\begin{align*}
\alpha < \mu_M(c), \\
\beta > \gamma_M(c), \\
\gamma < \mu_N(d), \\
\delta > \gamma_N(d),
\end{align*}
\]

this implies that

\[
\begin{align*}
\mu_M(d) = 0 & \quad \gamma_M(d) = 1, \\
\mu_N(c) = 0 & \quad \gamma_N(c) = 1.
\end{align*}
\]

Since for each \( x \in X \)

\[
\mu_N(x) \leq (1 - c - \beta)(x) = \begin{cases} 
\beta & \text{if } x = c, \\
1 & \text{if } x \neq c,
\end{cases}
\]

\[
\gamma_N(x) \geq c(\alpha)(x) = \begin{cases} 
\alpha & \text{if } x = c, \\
0 & \text{if } x \neq c,
\end{cases}
\]

we have that \( \mu_N \leq 1 - c - \beta, \gamma_N \geq c(\alpha) \), i.e. \( N \subseteq \overline{p} \).

On the other hand, does not exist a VIFP \( q \) such that \( q \in \overline{p} \), because if \( q = d(\gamma) \in \overline{p} \) we have \((1 - c - \beta)(d) = 0, \delta > c(\alpha)(d) \) and this yields a contradiction for every \( d \in X \).

The previous lemma gives that \( \overline{p} \) is IFOS, i.e. \( p \) is IFCS in \((X, \tau)\). \( \square \)
**Corollary 1.** Let \((X, \tau)\) be a \(T_2\) IFTS, if \((X, \tau)\) is normal, then it is also a regular IFTS.

**Definition 7.** (see [12]) We call a fuzzy topological space \((X, \tau)\) weakly normal if for every two closed sets \(\nu_1\) and \(\nu_2\) such that \(\nu_1 \land \nu_2 = \emptyset\), there exists two open sets \(\mu_1\) and \(\mu_2\) such that \(\nu_i \leq \mu_i\ (i=1,2)\) and \(\mu_1\) is not quasi-coincident with \(\mu_2\).

**Definition 8.** (see [14]) A fuzzy topological space \((X, T)\) is said para-regular if for every non-void closed set \(\nu\) and every fuzzy point \(x\) such that \(x /\in \text{supp}\ \nu\) there exist two open sets \(\mu_1\) and \(\mu_2\) such that \(\nu \leq \mu_1, x_\lambda \leq \mu_2\) and \(\mu_1 \land \mu_2 = \emptyset\).

**Proposition 2.** If \((X, \tau)\) is a regular IFTS then \((X, \tau_1)\) is a para-regular fuzzy topological space (where \(\tau_1 = \{\mu_A | A \in \tau\}\)).

**Proof.** Let \(\nu\) a non-void closed set in \((X, \tau_1)\) and \(z_\lambda\) a fuzzy point such that \(z_\lambda \land \text{supp} \ \nu = 0\) (i.e. \(\nu(z) = 0\)).

Let \(p = z(\lambda, 1 - \lambda)\) and \(C = < x, \nu, 1 - \nu >\), then \(C\) is an IFCS in \((X, \tau)\).

We have that \((z_\lambda \land \nu)(x) = 0\) for all \(x \in X\) and \((1 - z_\lambda) \lor (1 - \nu))(x) = 1\) for all \(x \in X\), then \(p \cap C = 0_\infty\).

From the hypothesis it follows that there exist two IFOSs \(M\) and \(N\) such that \(p \subseteq M, C \subseteq N\) and \(M \cap N = 0_\infty\), then \(\mu_M, \mu_N \in \tau_1, \mu_M \land \mu_N = \emptyset, z_\lambda \leq \mu_M\) and \(\nu \leq \mu_N\). \(\square\)

**Proposition 3.** If \((X, \tau)\) is a normal IFTS then \((X, \tau_1)\) is a weakly normal fuzzy topological space.

**Proof.** For every two closed fuzzy sets \(\nu_1\) and \(\nu_2\) in \((X, \tau_1)\) such that \(\nu_1 \land \nu_2 = \emptyset\), the IFCSs \(C_i = < x, \nu_i, 1 - \nu_i >\ (i=1,2)\) verify that \(C_1 \cap C_2 = 0_\infty\), then there exist two IFOSs \(M_i\) \((i=1,2)\) such that \(C_i \subseteq M_i\) \((i=1,2)\) and \(M_1 \cap M_2 = 0_\infty\). Finally, \(\mu_{M_i} \in \tau_1\) \((i=1,2)\), \(\mu_{M_1} \land \mu_{M_2} = \emptyset\), \(\nu_i \leq \mu_{M_i}\) \((i=1,2)\), then \(\mu_{M_1}\) is not quasi-coincident with \(\mu_{M_2}\). \(\square\)

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References


