ON PERTURBATION PROPERTIES OF FUZZY
RELATION EQUATIONS WITH
MAX-PRODUCT COMPOSITION

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Abstract: In this paper, the solving process of fuzzy relation equations with max-product composition is simplified. By the fuzzy solution invariant matrices, the perturbation properties of fuzzy relation equations with max-product composition are considered.

AMS Subject Classification: 15A09
Key Words: fuzzy relation equation, solution invariant matrices, perturbation

* The perturbation techniques is an important mathematical tool and the perturbation theory of fuzzy relation equations to be applied to fuzzy control, fuzzy inference and fuzzy logic as well. Tang [6] has discussed the perturbation issues of fuzzy relation equations with max-min composition. In this paper, the solution invariant matrix of fuzzy matrix is defined and the solving process of
fuzzy relation equations with max-product composition is simplified. By the fuzzy solution invariant matrices, the perturbation properties of fuzzy relation equations are considered.

1. Basic Notions

Let

\[ A \circ X = B, \quad \bigvee_{j=1}^{n} (a_{ij} x_j) = b_i \quad (i = 1, 2, \ldots, m) \]

be a fuzzy relation equation, where

\[ A = (a_{ij})_{n \times m}, \quad X = (x_j)_{n \times 1}, \quad B = (b_i)_{m \times 1} \]

are fuzzy matrices with elements belongs to \([0, 1]\) and the sign “\(\circ\)” stands for the max-product composition.

Similar to the paper [6], the perturbation elements in matrix \(A\) can be defined as follows.

**Definition 1.1.** In equation (1), assume that \(a_{ij}\) is an element of \(A\). If for \(\varepsilon > 0\), where \(\varepsilon\) is small enough, such that \(a_{ij} - \varepsilon > 0, a_{ij} + \varepsilon \leq 1\), when \(a_{ij}\) perturbs in \([a_{ij} - \varepsilon, a_{ij} + \varepsilon]\), the set of all solution of the equation (1) varies, then \(a_{ij}\) is called an element without perturbation in \(A\), denoted by EWP.

**Definition 1.2.** In equation (1), assume that \(a_{ij}\) is an element of \(A\). If \(a_{ij}\) perturbs within \([a_{ij}', 1]\)(\(a_{ij} \leq a_{ij}'\)), the set of all solution of the equation (1) is invariable, then \(a_{ij}\) is called an upper perturbation element in \(A\), denoted by UPE. If \(a_{ij}\) perturbs within \([0, a_{ij}']\)(\(a_{ij} \leq a_{ij}'\)), the set of all solution of the equation (1) is invariable, then \(a_{ij}\) is called a lower perturbation element in \(A\), denoted by LPE. If \(a_{ij}\) perturbs within \([b, a_{ij}']\)(\(b \leq a_{ij} \leq a_{ij}'\)), or \(a_{ij}, b\) \(a_{ij} \leq a_{ij} \leq b\)), the set of all solution of the equation (1) is invariable, then \(a_{ij}\) is called a lower-closed middle perturbation element in \(A\), denoted by LCMPE. If \(a_{ij}\) perturbs within \([0, 1]\), the set of all solution of the equation (1) is invariable, then \(a_{ij}\) is called a full perturbation element in \(A\), denoted by FPE.

**Definition 1.3.** In equation (1), assume that \(a_{ij}\) is an element of \(A\). If \(a_{ij}\) perturbs within \([a_{ij}', 1]\)(\(a_{ij} \geq a_{ij}'\)), the set of all solution of the equation (1) is invariable, then \(a_{ij}\) is called an upper-closed perturbation element in \(A\), denoted by UCPE. If \(a_{ij}\) perturbs within \([0, a_{ij}']\)(\(a_{ij} \leq a_{ij}'\)), the set of all solution of the
equation (1) is invariable, then $a_{ij}$ is called a lower-closed perturbation element in $A$, denoted by LCPE. If $a_{ij}$ is not only an LCMPE but also a UCMPE, then is called a MCPE.

2. Fuzzy Solution Invariant Matrix

The solving process of fuzzy relation equation

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\circ
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
=
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix}
$$

has been studied by some scholars [5], [3], [2], [4], [1]. We have given the following simplifying by the fuzzy solution invariant matrix.

**Definition 2.1.** Let $\beta : [0, 1]^2 \rightarrow [0, 1]$ be a mapping, $\forall a, b \in [0, 1]$, define

$$a\beta b = \begin{cases}
  b/a, & \text{if } a > b, \\
  1, & \text{if } a \leq b.
\end{cases}$$

For the equation (1), we have the following result.

**Lemma 2.1.** If the equation (1) is solvable, then

$$\bigwedge_{j=1}^{n} a_{ij} \geq b_i, \quad i = 1, 2, \cdots, m.$$  

**Lemma 2.2.** The equation (1) is solvable iff $X_d = A^T \beta B$ is the maximum solution of the equation (1), where $X_d = (c_1, c_2, \cdots, c_n)^T$, and $c_j = \bigwedge_{i=1}^{m} a_{ij} b_i$.

**Definition 2.2.** If the equation (1) is solvable, then $S(A, B) = X_0 | A \circ X_0 = B$ is called the set of all solution of the equation (1). For $X_1, X_2 \in S(A, B)$, $X_i = (x_1^i, x_2^i, \cdots, x_n^i)(i = 1, 2$), let $X_1 \leq X_2$ iff $x_k^1 \leq x_k^2(k = 1, 2, \cdots, n)$. It is obvious that $\leq$ is a partial ordering on $S(A, B)$, $(S(A, B), \leq)$ is a lattice with min and max as its meet and join, respectively.

In the following, we suppose $S(A, B) \neq \emptyset$.

**Definition 2.3.** Given the equation (1), then

$$A^{(1)} = (a^{(1)}_{ij}), a^{(1)}_{ij} = \begin{cases}
  a_{ij}, & a_{ij} c_j = b_i, \\
  0, & \text{otherwise}.
\end{cases}$$
is called the reduced matrix of A, where \( X_d = (c_1, c_2, \cdots, c_n)^T \) is the maximum solution of the equation (1).

**Theorem 2.3.** Given the equation (1) then

\[ S(A, B) = S(A^{(1)}, B). \]

**Proof.** Suppose that \( X_d = (c_1, c_2, \cdots, c_n)^T \) is the maximum solution of the equation (1). \( \forall i \in \{1, 2, \cdots, m\} \), we denote \( I_1^i = \{ j | a_{ij}c_j < b_i \} \), \( I_2^i = \{ j | a_{ij}c_j = b_i \} \), \( I_3^i = \{ j | a_{ij}c_j > b_i \} \). For brevity, \( I_1^i, I_2^i, I_3^i \) are denoted by \( I_1, I_2, I_3 \) respectively. If there exists a pair \((i, j)\) such that \( a_{ij}c_j > b_i \), then \( \bigvee_{j=1}^{n} (a_{ij}c_j) \geq b_i \), it implies \( I_3 = \emptyset \).

Let \( X = (x_1, x_2, \cdots, x_n)^T \in S(A, B) \), then for all \( i \), we have

\[ b_i = \bigvee_{j=1}^{n} (a_{ij}x_j) = (\bigvee_{j \in I_1} (a_{ij}x_j)) \bigvee (\bigvee_{j \in I_2} (a_{ij}x_j)). \]

Since \( \bigvee_{j \in I_1} (a_{ij}x_j) < b_i \), thus \( b_i = \bigvee_{j \in I_1} (a_{ij}x_j) = \bigvee_{j=1}^{n} (a_{ij}^{(1)}x_j). \) Then \( X = (x_1, x_2, \cdots, x_n)^T \in S(A^{(1)}, B). \)

Conversely, suppose that \( J_1 = \{ j | a_{ij}^{(1)} = 0 \} \), \( J_2 = \{ j | a_{ij}^{(1)} \neq 0 \} \), then \( J_1 = I_1 \cup I_3 = I_1, I_2 = I_2, \) hence

\[ c_j = \bigwedge_{i=1}^{m} (a_{ij}\beta b_i) = (\bigwedge_{j \in J_1} (a_{ij}\beta b_i)) \bigwedge (\bigwedge_{j \in J_2} (a_{ij}\beta b_i)). \]

If \( j \in J_1 \), and \( a_{ij} \leq b_i \), then \( a_{ij}\beta b_i = a_{ij}^{(1)}\beta b_i = 1. \)

If \( j \in J_1, a_{ij} > b_i \), and \( a_{ij}c_j \neq b_i \), then \( a_{ij}c_j < b_i, \ a_{ij}\beta b_i = b_i/a_{ij} > c_j \), thus

\[ c_j = \bigwedge_{j \in J_2} (a_{ij}\beta b_i) = \bigwedge_{i=1}^{m} (a_{ij}^{(1)}\beta b_i). \]

It illustrates that the maximum solution of \( A \circ X = B \) is equal to the ones of \( A^{(1)} \circ X = B \).

Suppose \( X = (x_1, x_2, \cdots, x_n)^T \in S(A^{(1)}, B) \), we have

\[ b_i = \bigvee_{j=1}^{n} (a_{ij}^{(1)}x_j) \leq \bigvee_{j=1}^{n} (a_{ij}x_j) \leq \bigvee_{j=1}^{n} (a_{ij}c_j) = b_i; \]
then \(X \in S(A, B)\), it implies that \(S(A, B) = S(A^{(1)}, B)\).

**Definition 2.4.** Let \(A \circ X = B\) and \(E \circ X = B\) be two fuzzy relation equations. \(A\) and \(E\) are called the fuzzy solution invariant matrices about \(B\) if \(S(A, B) = S(E, B)\).

**Theorem 2.4.** Let \(A \circ X = B\) and \(E \circ X = B\) be two fuzzy relation equations. \(A\) and \(E\) are fuzzy solution invariant matrices about \(B\) iff \(S(A^{(1)}, B) = S(E^{(1)}, B)\).

Proof. It is clear by Theorem 2.3 and Definition 2.4.

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3. The Perturbation Issues of Fuzzy Relation Equations

**Definition 3.1.** Let \(A = (a_{ij})_{m \times n}\) and \(C = (c_{ij})_{m \times n}\) be two fuzzy matrices. We shall write \(A \leq C\) if \(a_{ij} \leq c_{ij}\) for all pairs \((i, j)\), where \(1 \leq i \leq m, 1 \leq j \leq n\). And write \(A < C\) if \(A \leq C\) and \(A \neq B\).

**Definition 3.2.** Given the equation (1). \(A\) is called a fuzzy matrix without perturbation about \(B\) if every element \(a_{ij}\) of \(A\) is the EWP of \(A\). Otherwise, \(A\) is called a fuzzy perturbation matrix.

**Theorem 3.1.** Given the equation (1). Then \(a_{ij}\) is a LPE of \(A\) within \([0, b_i]\) if \(a_{ij} < b_i\). Moreover, \(a_{ij}\) is a LCPE of \(A\) within \([0, b_i]\) when \(c_j < 1\). \(a_{ij}\) is a LPE of \(A\) within \([0, b_i]\) and is not a LCPE of \(A\) within \([0, b_i]\) when \(c_j = 1\).

Proof. If \(a_{ij} < b_i\), then \(a_{ij} \beta b_i = 1\) and \(a_{ij}^{(1)} = 0\). We replace \(a_{ij}\) by \(e_{ij}(e_{ij} \in [0, b_i])\) and let the other elements are invariable, so we get a new matrix \(E = (e_{ij})\). Hence \(e_{ij} \beta b_i = 1\) and \(e_{ij}^{(1)} = 0\), thus \(E^T \beta B = A^T \beta B\) and \(E^{(1)} = A^{(1)}\). From Theorem 2.4, \(S(E, B) = S(A, B)\), \(a_{ij}\) is a LPE of \(A\) within \([0, b_i]\).

Moreover, when \(c_j < 1\), for above the matrix \(E\), if \(e_{ij} = b_i\), then \(e_{ij} \beta b_i = 1\) and \(e_{ij}^{(1)} = 0\). Thus \(E^T \beta B = A^T \beta B\) and \(E^{(1)} = A^{(1)}\). \(a_{ij}\) is a LCPE of \(A\) within \([0, b_i]\).

When \(c_j = 1\), \(e_{ij}^{(1)} = b_i < 1\), it implies that \(E^{(1)} \neq A^{(1)}\), so \(S(E, B) \neq S(A, B)\), \(a_{ij}\) is not a LCPE of \(A\) within \([0, b_i]\).

**Theorem 3.2.** Given the equation (1). If \(a_{ij} b_i\) and \(a_{ij}^{(1)} = 0\), then \(a_{ij}\) is a LCPE of within \([0, b_i]\).

Proof. Assume that \(a_{ij} b_i\) and \(a_{ij}^{(1)} = 0\), then \(a_{ij} c_j < b_i\), so \(c_j < 1\). We replace \(a_{ij}\) by \(e_{ij}(e_{ij} \in [0, b_i])\) and let the other elements be invariable, so we get a new matrix \(E = (e_{ij})\). Then \(e_{ij} \beta b_i = a_{ij} \beta b_i = 1\) and \(e_{ij}^{(1)} = a_{ij}^{(1)} = 0\), thus
$ET\beta B = A^T\beta B$ and $E^{(1)} = A^{(1)}$, it implies that $S(E, B) = S(A, B)$, $a_{ij}$ is a LCPE of $A$ within $[0, b_i]$. 

**Theorem 3.3.** Given the equation (1) and assume that $a_{ij} > b_i$ and $a_{ij}^{(1)} = 0$, then $a_{ij}$ is a LCMPE of $A$ within $[b_i, b_i/c_j]$. 

Proof. Assume that $a_{ij} > b_i$ and $a_{ij}^{(1)} = 0$, then $b_i/a_{ij} = a_{ij}\beta b_i > \bigwedge_{t=1}^{m} (a_{tij}\beta b_t) = c_j$. We replace $a_{ij}$ by $e_{ij}([b_i, b_i/c_j])$, and let the other elements be invariable, so we get a new matrix $E = (e_{ij})$. Then $b_i/e_{ij} \in (c_j, 1]$, $b_i/e_{ij} = e_{ij}\beta b_i > c_j$. Hence

$$c_j = \bigwedge_{t=1}^{m} a_{tij}\beta b_t = (\bigwedge_{t\neq i} a_{tij}\beta b_t) \bigwedge (a_{ij}\beta b_i) = \bigwedge_{t\neq i} a_{tij}\beta b_t$$

$$= (\bigwedge_{t\neq i} a_{tij}\beta b_t) \bigwedge (e_{ij}\beta b_i) = (\bigwedge_{t\neq i} e_{tij}\beta b_t) \bigwedge (e_{ij}\beta b_i) = \bigwedge_{t=1}^{m} e_{tij}\beta b_t.$$ 

So $A^T\beta B = E^T\beta B$. Moreover, $e_{ij}^{(1)} = 0 = a_{ij}^{(1)}$, $E^{(1)} = A^{(1)}$, by Theorem 2.4, $S(E, B) = S(A, B)$, $a_{ij}$ is a LCMPE of $A$ within $[b_i, b_i/c_j]$.

**Theorem 3.4.** Given the equation (1), if $a_{ij}^{(1)} \neq 0$, then $a_{ij}$ is an EWP of $A$.

Proof. Suppose $a_{ij}^{(1)} \neq 0$, then $a_{ij}c_j = b_i$,

$$c_j = b_i/a_{ij} = a_{ij}\beta b_i = (\bigwedge_{t\neq i} a_{tij}\beta b_t) \bigwedge (a_{ij}\beta b_i),$$

thus $a_{ij}\beta b_i = (\bigwedge_{t\neq i} a_{tij}\beta b_t)$. If we replace $a_{ij}$ by $e_{ij}([0, b_i] \cup [b_i/c_j, 1])$, and let the other elements be invariable, so we get a new matrix $E = (e_{ij})$. Then $e_{ij}^{(1)} = 0 \neq a_{ij}^{(1)}$, $E^{(1)} \neq A^{(1)}$ it implies that $S(E, B) \neq S(A, B)$.

Thus $a_{ij}$ is an EWP of $A$.

**Corollary 3.5.** Let $a_{ij}^{(1)} \neq 0$ and $a_{ij}$ be an EWP of $A$, if $a_{ij}$ perturbs within $[0, b_i)$ and let the other elements are invariable, so we get a new matrix $E$. Then $S(E, B) \subset S(A, B)$.

**Corollary 3.6.** Let $a_{ij}^{(1)} \neq 0$ and $a_{ij}$ be an EWP of $A$, if $a_{ij}$ perturbs within $(b_i/c_j, 1]$ and let the other elements are invariable, so we get a new matrix $E$. Then $S(E, B) \subset S(A, B)$.

**Theorem 3.7.** Given the equation (1), $A$ is a fuzzy matrix without perturbation about $B$ if and only if $A = A^{(1)}$. 

Acknowledgements

This project is supported by the NNSF of P.R. China (60364001) and NSF of Henan (200310011).

References


