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LINEAR INITIAL-VALUE PROBLEMS COUNTABLY DETERMINED

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Abstract: In this paper we approximate the solution of a linear initial-value problem, making use of a Schauder basis for certain Banach space associated with such a differential problem.

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1. Introduction

In this work some results are discussed in order to approximate the solution of the following initial-value problem: given $x_0 \in \mathbb{R}^n$, $a \in C([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))$ ($\mathcal{M}_n(\mathbb{R})$ is the set of all $n \times n$ real matrices) and $b \in C([\alpha, \alpha+\beta], \mathbb{R}^n)$, find $x \in C^1([\alpha, \alpha+\beta], \mathbb{R}^n)$ such that

$$\left\{ \begin{array}{ll} x'(t) = a(t)x(t) + b(t), & t \in [\alpha, \alpha+\beta]\,, \\ x(\alpha) = x_0\,. \end{array} \right. \tag{1.1}$$

The key idea is to combine classical techniques (fixed point) with certain properties of Schauder bases. Thus we do not need to solve systems of algebraical

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linear equations – collocation methods – or to use quadrature formulas.

Let us recall (Megginson [3]) that a sequence $\{x_j\}_{j\geq 1}$ in a Banach space X is said to be a *Schauder basis* provided that for all $x \in X$ there exists a unique sequence of scalars $\{\lambda_j\}_{j\geq 1}$ in such a way that $x = \sum_{j\geq 1} \lambda_j x_j$. The j-th (continuous and linear) *biorthogonal functional* x_j^* is defined at such an x as $x_j^*(x) = \lambda_j$, and the j-th (continuous and linear) *projection* Q_j by $Q_j(x) = \sum_{i=1}^{j} \lambda_i x_i$.

Starting in Section 2 with a background about the classical Schauder basis in the Banach space $C([\alpha, \alpha+\beta]) = C([\alpha, \alpha+\beta], \mathbb{R})$ and some fixed point questions for the operator related to the initial-value problem considered, we get in Section 3 to an operative expression of such an operator in terms of certain linear combinations of evaluations of the data functions. Such expression enables us to derive, with the Banach Fixed Point Theorem, a result (Theorem 3) to approximate the solution of the initial-value problem. A numerical example in Section 4 illustrates the preceding results.

2. Preliminaries

We begin by introducing the classical Schauder basis for the space $C([\alpha, \alpha+\beta])$, endowed with its usual sup-norm. Suppose that $\{t_j\}_{j\geq 1}$ is a dense sequence of distinct points in $[\alpha, \alpha+\beta]$ such that $t_1 = \alpha$ and $t_2 = \alpha + \beta$. The classical Schauder basis $\{\Gamma_j\}_{j\geq 1}$ (associated with $\{t_j\}_{j\geq 1}$) for the Banach space $C([\alpha, \alpha+\beta])$ is defined as follows:

$$\Gamma_1(t) = 1 \quad (\alpha \leq t \leq \alpha+\beta)$$

and for all $j > 1$, Γ_j is the piecewise linear continuous function with nodes at $t_1, \ldots, t_j$, such that

$$\text{for all } 1 \leq i < j, \quad \Gamma_j(t_i) = 0\,,$$

while

$$\Gamma_j(t_j) = 1.$$

In what follows, $\{\Gamma_j\}_{j\geq 1}$ will denote such basis and $\{\Gamma_j^*\}_{j\geq 1}$ and $\{Q_j\}_{j\geq 1}$, respectively, the associated sequences of biorthogonal functionals and projections. In the next statement we collect some basic elementary facts that will play a fundamental role in our results. For a proof, see Megginson [3] or Semadeni [4].

Proposition 1. *Let $x \in C([\alpha, \alpha+\beta])$. Then*

$$\Gamma_1^*(x) = x(t_1) \tag{2.1}$$

and for all $j > 1$,

$$\Gamma_j^*(x) = x(t_j) - \sum_{i=1}^{j-1} \Gamma_i^*(x)\Gamma_i(t_j). \tag{2.2}$$

In particular, for all $j \geq 1$ *and for all* $i \leq j$,

$$(Q_j x)(t_i) = x(t_i). \tag{2.3}$$

Another tool that we shall use in the following is the Banach Fixed Point Theorem in the following form (Jamenson [2]): let T be a self-mapping of a (nonempty) Banach space $(X, \|\cdot\|)$ and let $\{\lambda_j\}_{j\geq 1}$ be a sequence of non-negative real numbers such that the series $\sum_{j\geq 1} \lambda_j$ is convergent and for all $x, y \in X$ and for all $j \geq 1$, $\|T^j x - T^j y\| \leq \lambda_j \|x - y\|$. Then T has a unique fixed point $u \in X$. Moreover, if $\bar{x}$ is an element in X then $u = \lim_j T^j(\bar{x})$. In fact, we have that for all $j \geq 1$,

$$\|T^j \bar{x} - u\| \leq \left(\sum_{i=j}^{\infty} \lambda_i \right) \|T\bar{x} - \bar{x}\|.$$

Let us now consider the operator

$$T : C([\alpha, \alpha + \beta], \mathbb{R}^n) \longrightarrow C([\alpha, \alpha + \beta], \mathbb{R}^n),$$

associated with the initial-value problem (1.1), defined by

$$(Tx)(t) := x_0 + \int_\alpha^t (a(s)x(s) + b(s))ds,$$

$$(t \in [\alpha, \alpha + \beta],\ x \in C([\alpha, \alpha + \beta], \mathbb{R}^n)), \tag{2.4}$$

where the norm in the space $C([\alpha, \alpha + \beta], \mathbb{R}^n)$ is the sup-sup one:

$$\|x\|_\infty := \sup_{t \in [\alpha, \alpha+\beta]} \|x(t)\|_\infty, \qquad (x \in C([\alpha, \alpha + \beta], \mathbb{R}^n)).$$

Since T satisfies that for all $x, y \in C([\alpha, \alpha + \beta], \mathbb{R}^n)$ and for all $j \geq 1$,

$$\|T^j x - T^j y\|_\infty \leq \frac{1}{j!}(M\beta)^j \|x - y\|_\infty, \tag{2.5}$$

with $M := \max_{\alpha \leq t \leq \alpha+\beta} \|a(t)\|_\infty$, then it follows from the Banach Fixed Point Theorem and the convergence of the series $\sum_{j\geq 1} \frac{(M\beta)^j}{j!}$ for any β and M, that for each $\bar{x}$ in $C([\alpha, \alpha+\beta], \mathbb{R}^n)$, the sequences $\{T^j \bar{x}\}_{j\geq 1}$ in $C([\alpha, \alpha+\beta], \mathbb{R}^n)$ converges uniformly to the unique solution u of the initial-value problem (1.1) (the fixed point of T). Moreover, as a consequence of inequality (2.5) and Taylor's formula we have the following estimate of the rate of convergence: for all $j \geq 1$,

$$\|T^j \bar{x} - u\|_\infty \leq \frac{(M\beta)^j}{j!} e^{M\beta} \|T\bar{x} - \bar{x}\|_\infty. \tag{2.6}$$

3. Schauder Bases and Linear Initial-Value Problems

The next result enables to obtain the image under operator T of any continuous function in terms of certain sequences of scalars, sequences which are obtained just by evaluating some functions at adequate points. We shall consider the sup-sup norm on the space $C([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))$:

$$\|a\|_\infty := \sup_{t\in[\alpha,\alpha+\beta]} \|a(t)\|_\infty, \qquad (a \in C([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))).$$

Theorem 2. *Let $n \geq 1$ and assume that $a = (a_{ij})_{i,j=1,\ldots n} \in C([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))$, $b = (b_j)_{j=1,\ldots,n} \in C([\alpha, \alpha+\beta], \mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. Given $1 \leq j, k \leq n$ let $\{a_{jk}^{(i)}\}_{i\geq 1}$ and $\{b_j^{(i)}\}_{i\geq 1}$ be the sequences of scalars satisfying*

$$a_{jk} = \sum_{i\geq 1} a_{jk}^{(i)} \Gamma_i \qquad \text{and} \qquad b_j = \sum_{i\geq 1} b_j^{(i)} \Gamma_i. \tag{3.1}$$

Let us consider the continuous integral operator

$$T : C([\alpha, \alpha+\beta], \mathbb{R}^n) \longrightarrow C([\alpha, \alpha+\beta], \mathbb{R}^n)$$

defined in (2.4). Then, for all $x = (x_j)_{j=1,\ldots,n} \in C([\alpha, \alpha+\beta], \mathbb{R}^n)$ and for all $t \in [\alpha, \alpha+\beta]$ we have that

$$(Tx)(t) = x_0 + \left(\sum_{i\geq 1} c_j^{(i)} \int_\alpha^t \Gamma_i(s) ds \right)_{j=1,\ldots,n},$$

where for $j = 1, \ldots, n$,

$$\begin{cases} c_j^{(1)} = & b_j^{(1)} + \sum_{k=1}^n a_{jk}^{(1)} x_k(t_1), \\ c_j^{(i)} = & \sum_{l=1}^i \left(b_j^{(l)} + \sum_{k=1}^n a_{jk}^{(l)} x_k(t_i) \right) \Gamma_k(t_i) - \sum_{l=1}^{i-1} c_j^{(l)} \Gamma_l(t_i), \quad \text{if } i \geq 2. \end{cases}$$

Proof. Let us start by pointing out that we can assure the existence of the sequences $\{a_{jk}^{(i)}\}_{i\geq 1}$ and $\{b_j^{(i)}\}_{i\geq 1}$, because the functions a_{jk} and b_j are continuous. Let us fix a continuous function $x = (x_j)_{j=1,\ldots,n} \in C([\alpha, \alpha+\beta], \mathbb{R}^n)$. Since for all $t \in [\alpha, \alpha+\beta]$ we have that

$$(Tx)'(t) = a(t)x(t) + b(t),$$

then taking into account (3.1) one arrives at

$$\begin{aligned}(Tx)'(t) &= \left(b_j(t) + \sum_{k=1}^{n} a_{jk}(t)x_k(t)\right)_{j=1,\ldots,n} \\ &= \left(\sum_{i\geq 1} b_j^{(i)}\Gamma_i(t) + \sum_{k=1}^{n}\left(\sum_{i\geq 1} a_{jk}^{(i)} x_k(t)\Gamma_i(t)\right)\right)_{j=1,\ldots,n} \\ &= \left(\sum_{i\geq 1}\left(b_j^{(i)} + \sum_{k=1}^{n} a_{jk}^{(i)} x_k(t)\right)\Gamma_i(t)\right)_{j=1,\ldots,n} = \left(\sum_{i\geq 1} c_j^{(i)}\Gamma_i(t)\right)_{j=1,\ldots,n},\end{aligned}$$

where the sequence of scalars $\{c_j^{(i)}\}_{i\geq 1}$, defined as above, is derived from (2.1) and (2.2). Therefore, by integrating the preceding expression one has that

$$(Tx)(t) = x_0 + \left(\sum_{i\geq 1} c_j^{(i)} \int_{\alpha}^{t} \Gamma_i(s)ds\right)_{j=1,\ldots,n},$$

as required. □

Theorem 3. *Let $n \geq 1$ and assume that $a = (a_{ij})_{i,j=1,\ldots n} \in C([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))$, $b = (b_j)_{j=1,\ldots,n} \in C([\alpha, \alpha+\beta], \mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. Let $T : C([\alpha, \alpha+\beta], \mathbb{R}^n) \longrightarrow C([\alpha, \alpha+\beta], \mathbb{R}^n)$ be the operator defined in (2.4). Let $\bar{x} : [\alpha, \alpha+\beta] \longrightarrow \mathbb{R}^n$ be a continuous function and $m \geq 1$ and $n_1, \ldots, n_m \geq 1$. Consider the continuous function*

$$y_0(t) := \bar{x}(t) \quad (t \in [\alpha, \alpha+\beta])$$

and for $r = 1, \ldots, m$ the continuous functions

$$L_{r-1}(t) := a(t)y_{r-1}(t) + b(t) \quad (t \in [\alpha, \alpha+\beta]),$$

and

$$y_r(t) := x_0 + \int_{\alpha}^{t} (Q_{n_r}(L_{r-1}(s))_k)_{k=1,\ldots,n} ds \quad (t \in [\alpha, \alpha+\beta]).$$

Assume in addition that certain positive numbers $\varepsilon_1, \dots, \varepsilon_m$ *satisfy*

$$\|Ty_{r-1} - y_r\|_\infty < \varepsilon_r.$$

Then, if u *is the solution of the linear initial-value problem (1.1), then we have that*

$$\|u - y_m\|_\infty \leq \frac{(M\beta)^m}{m!} e^{M\beta} \|T\bar{x} - \bar{x}\|_\infty + \sum_{r=1}^{m} \varepsilon_r \frac{(M\beta)^{m-r}}{(m-r)!},$$

where $M = \max_{\alpha \leq t \leq \alpha+\beta} \|a(t)\|_\infty$.

Proof. Since

$$\|u - y_m\|_\infty \leq \|u - T^m\bar{x}\|_\infty + \|y_m - T^m\bar{x}\|_\infty, \tag{3.2}$$

we shall separately obtain upper bounds for both terms on the left hand side in (3.2). On the one hand, inequality (2.6) gives

$$\|u - T^m\bar{x}\|_\infty \leq \frac{(M\beta)^m}{m!} e^{M\beta} \|T\bar{x} - \bar{x}\|_\infty. \tag{3.3}$$

On the other hand, since

for all $v, w \in C([\alpha, \alpha+\beta], \mathbb{R}^n)$ and for all $j \geq 1$,

$$\|T^j v - T^j w\|_\infty \leq \frac{1}{j!} (M\beta)^j \|v - w\|_\infty,$$

then for all $r = 1, \dots, m$ we have that

$$\begin{aligned} \|T^{m-r+1} y_{r-1} - T^{m-r} y_r\|_\infty &= \|T^{m-r}(Ty_{r-1}) - T^{m-r} y_r\|_\infty \\ &\leq \frac{1}{(m-r)!} (M\beta)^{m-r} \|Ty_{r-1} - y_r\|_\infty, \end{aligned}$$

and hence

$$\begin{aligned} \|y_m - T^m\bar{x}\|_\infty = \|y_m - T^m y_0\|_\infty &\leq \sum_{r=1}^{m} \|T^{m-r+1} y_{r-1} - T^{m-r} y_r\|_\infty \\ &\leq \sum_{r=1}^{m} \varepsilon_r \frac{(M\beta)^{m-r}}{(m-r)!}. \end{aligned} \tag{3.4}$$

Finally, the proof is complete in view of (3.2), (3.3) and (3.4). □

Note that given $\varepsilon_1, \dots, \varepsilon_m > 0$ we can find positive integers $n_1, \dots, n_m$ such that $\|Ty_{r-1} - y_r\|_\infty < \varepsilon_r$, since for all $x \in C([\alpha, \alpha+\beta])$, $\lim_{j \geq 1} \|Q_j x - x\|_\infty = 0$.

However, if we wish to find the integers $m, n_1, \dots, n_m$ from the positive numbers $\varepsilon_1, \dots, \varepsilon_m$, we can use this easy and well-known consequence of the Mean Value Theorem and the interpolating property (2.3) of the basis for $C([\alpha, \alpha+\beta])$: suppose that $x \in C^1([\alpha, \alpha+\beta])$ (in fact, we can assume that x is a continuous and C^1 class function on $[\alpha, \alpha+\beta]$, except perhaps for a finite number of points), $j \geq 2$ and

$$h := \max_{i=2,\dots,j} (s_i - s_{i-1}),$$

where $\{s_1 = \alpha < s_2 < \cdots < s_{j-1} < s_j = \alpha+\beta\}$ is the set $\{t_1, \dots, t_j\}$ ordered in a increasing way (if $\|x'\|_\infty = 0$ there is no additional hypothesis on the nodes). Then

$$\|x - Q_j x\|_\infty \leq 2\|x'\|_\infty h. \tag{3.5}$$

If one assumes in the initial-value problem that a and b are functions of C^1 class on $[\alpha, \alpha+\beta]$ then the norm appearing in Theorem 3, $\|Ty_{r-1} - y_r\|_\infty$ can be estimated as follows:

$$\|Ty_{r-1} - y_r\|_\infty \leq \beta \|L_r - (Q_{n_r}(L_r)_k)_{k=1,\dots,n}\|_\infty$$

and then above applies. These ideas are developed more precisely in the following results.

Lemma 4. *The sequence $\{L_r'\}_{r\geq 1}$ is uniformly bounded, provided that a and b are C^1-class functions.*

Proof. It is a well-known fact (Megginson [3]) that the classical Schauder basis for $C[\alpha, \alpha+\beta]$ is monotone, i.e., for all $j \geq 1$, $\|Q_j\| = 1$. Then, for all $r \geq 1$,

$$\|y_r(t)\| \leq \|x_0\| + \int_\alpha^t \|L_{r-1}(s)\| ds \leq \|x_0\| + (t-\alpha)(\|a\|\|y_{r-1}\| + \|b\|).$$

An easy recursive argument gives us that for all $r \geq 1$ and for all $t \in [\alpha, \alpha+\beta]$ it holds

$$\|y_r(t)\| \leq \|x_0\| \sum_{i=0}^{r-1} \|a\|^i (t-\alpha)^i + \|b\| \sum_{i=1}^{r} \|a\|^{i-1}(t-\alpha)^i + (t-\alpha)^r \|a\|^r \|y_0\|$$

and so

$$\|y_r\| \leq (\|x_0\| + \beta\|b\|) \sum_{i=0}^{r-1} (\|a\|\beta)^i + \beta^r \|a\|^r \|y_0\|,$$

which proves that the sequence $\{y_r\}_{r\geq 1}$ is uniformly bounded. On the other hand, since

$$y_r'(t) = Q_{n_r}(a(t)y_{r-1}(t) + b(t)),$$

then we arrive at the fact that the sequence $\{y'_r\}_{r\geq 1}$ also is bounded, because of the uniform boundness of $\{y_r\}_{r\geq 1}$ and the monotonicity of the classical Schauder basis. Finaly, the announced statement follows form the uniform boundness of the sequences $\{y_r\}_{r\geq 1}$ and $\{y'_r\}_{r\geq 1}$ and the equality

$$L'_r(t) = a'(t)y_{r-1} + a(t)y'_{r-1} + b'(t). \qquad \square$$

Theorem 5. *Assume that* $a = (a_{ij})_{ij=1,\ldots,n} \in C^1([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))$ *and* $b = (b_j)_{j=1,\ldots,n} \in C^1([\alpha, \alpha+\beta], \mathbb{R}^n)$. *Then there exists* $K > 0$ *such that for all* $m \geq 1$ *we have that*

$$\sum_{r=1}^{m} \|Ty_{r-1} - y_r\|_\infty \frac{(M\beta)^{m-r}}{(m-r)!} \leq Kh.$$

Proof. In view of (3.5), we have that for all $r \geq 1$

$$\|Ty_{r-1} - y_r\|_\infty \leq \beta \|L_r - (Q_{n_r}(L_r)_k)_{k=1,\ldots,n}\|_\infty \leq 2\beta \|L'_r\|_\infty h$$

and since the sequence $\{L'_r\}_{r\geq 1}$ is uniformly bounded (Lemma 4), then there exists $K_1 > 0$ such that for all $r \geq 1$ we have that

$$\|Ty_{r-1} - y_r\|_\infty \leq K_1 h.$$

Therefore, we deduce that

$$\sum_{r=1}^{m} \|Ty_{r-1} - y_r\|_\infty \frac{(M\beta)^{m-r}}{(m-r)!} \leq K_1 h e^{M\beta}.$$

Take $K := K_1 e^{M\beta}$ and the proof is complete. $\square$

Remark 6. Although our numerical method works for any Schauder basis in the Banach space $C([\alpha, \alpha+\beta])$, we have chosen the classical one because the biorthogonal functionals and the projections associated have an easy expression. Let us also point out that Schauder bases have been used as a fundamental tool in order to solve numerically some integral equations (Berenguer et al [1]).

4. A Numerical Example

Finally we exhibit an example which shows the behaviour of our results. To this end, we fix the data's initial-value problem: $x_0 \in \mathbb{R}^n$, $a = (a_{ij})_{ij=1,\dots,n} \in C^1([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))$ and $b = (b_j)_{j=1,\dots,n} \in C^1([\alpha, \alpha+\beta], \mathbb{R}^n)$ and take $\bar{x} = x_0$. We choose an $n \in \mathbb{N}$ with $n = 2^k + 1, k \in \mathbb{N}$, and thus

$$h = \max_{2 \le i \le n} (s_i - s_{i-1}) = \frac{1}{2^k}.$$

Then we calculate the sequences of coefficients $\{a_{jk}^{(i)}\}_{i=1}^n$ and $\{b_j^{(i)}\}_{i=1}^n$ and obtain recursively the functions y_r in Theorem 3, taking $n_1 = \cdots = n_r = n$ and $\bar{x} = x_0$. With such a choose for $\bar{x}$ we have that

$$T\bar{x}(t) - \bar{x}(t) = \int_\alpha^t (a(s)\bar{x}(s) + b(s))ds$$

and the norm $\|T\bar{x} - \bar{x}\|_\infty$ in Theorem 3 can be easily bounded. We also determine the errors

$$E_{nr} = \max_i |y_r(s_i) - u(s_i)|,$$

where u is the exact solution. We have considered the approximation of the exact solution y_m in such a way that

$$\left| \frac{E_{nm}}{E_{nm+1}} \right| < 1 + 10^{-2}.$$

Example. Consider the initial-value problem

$$\begin{cases} x'(t) = \begin{pmatrix} \frac{1}{t+2} & 0 \\ 3t+1 & \sqrt{t} \end{pmatrix} x(t) + \begin{pmatrix} -\frac{\sin t}{t+2} + \cos t \\ -\sin t - \sqrt{t}\cos t - (3t+1)\sin t \end{pmatrix}, \\ x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

whose exact solution is $u(t) = (\sin t, \cos t)$. In the columns of the following table we give the absolute errors E_{nm} in nine representative points of the approximations y_m, obtained with different values of n.

	$(n=9, m=7)$	$(n=17, m=8)$	$(n=33, m=8)$
0	$(0,0)$	$(0,0)$	$(0,0)$
0.125	$(1.67\times10^{-4}, 4.31\times10^{-6})$	$(4.18\times10^{-5}, 8\times10^{-7})$	$(1.04\times10^{-5}, 1.82\times10^{-7})$
0.250	$(3.41\times10^{-4}, 2.66\times10^{-5})$	$(8.53\times10^{-5}, 6.05\times10^{-6})$	$(2.13\times10^{-5}, 1.47\times10^{-6})$
0.375	$(5.19\times10^{-4}, 8.71\times10^{-5})$	$(1.29\times10^{-4}, 2.07\times10^{-5})$	$(3.24\times10^{-5}, 5.13\times10^{-6})$
0.500	$(6.98\times10^{-4}, 2.08\times10^{-4})$	$(1.74\times10^{-4}, 5.07\times10^{-5})$	$(4.36\times10^{-5}, 1.25\times10^{-5})$
0.625	$(8.73\times10^{-4}, 4.18\times10^{-4})$	$(2.18\times10^{-4}, 1.02\times10^{-4})$	$(5.46\times10^{-5}, 2.54\times10^{-5})$
0.750	$(1.04\times10^{-3}, 7.48\times10^{-4})$	$(2.61\times10^{-4}, 1.83\times10^{-4})$	$(6.52\times10^{-5}, 4.57\times10^{-5})$
0.875	$(1.20\times10^{-3}, 1.23\times10^{-3})$	$(3.01\times10^{-4}, 3.02\times10^{-4})$	$(7.53\times10^{-5}, 7.58\times10^{-5})$
1	$(1.35\times10^{-3}, 1.94\times10^{-3})$	$(3.39\times10^{-4}, 4.71\times10^{-4})$	$(8.48\times10^{-5}, 1.19\times10^{-4})$

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