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HIGHER INTEGRABILITY FOR VERY WEAK SOLUTIONS OF NONHOMOGENEOUS $\mathcal{A}$-HARMONIC EQUATION

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Abstract: This paper gives a higher integrability result for very weak solutions of the nonhomogeneous $\mathcal{A}$-harmonic equation

$$\mathrm{div}\mathcal{A}(x, g(x) + \nabla u(x)) = -f(x)$$

by using the results of Riesz transforms and interpolation.

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1. Introduction

Gehring's Lemma (cf [6]) and its many variants and extensions plays an important role in nonlinear PDEs, in the study of weighted norm inequalities for the classical operators of harmonic analysis, as well as in functional analysis. But in 2002, Capone, Greco and Iwaniec showed in [3] that the well established bounds for singular integrals are also sufficient to treat the higher integrability

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theory of the following nonlinear PDE

$$\operatorname{div} |g+\bigtriangledown u|^{p-2}(g+\bigtriangledown u) = \operatorname{div} h \tag{1.1}$$

by an elementary interpolation.

The aim of this paper is to obtain a similar result for the nonhomogeneous $\mathcal{A}$-harmonic equation

$$\operatorname{div}\mathcal{A}(x, g(x)+\nabla u(x)) = -f(x)\,, \tag{1.2}$$

which is somewhat different from equation (1.1). We assume that the operator $\mathcal{A}(\cdot,\cdot) : \mathrm{R}^n \to \mathrm{R}^n$ satisfies the following conditions:

(i) Lipschitz type condition

$$|\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta)| \le \beta|\xi-\zeta|(|\xi|+|\zeta|)^{p-2}\,.$$

(ii) Monotonicity condition

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta), \xi-\zeta\rangle \ge \alpha|\xi-\zeta|^2(|\xi|+|\zeta|)^{p-2}$$

for almost every $x \in \mathrm{R}^n$ and all $\xi,\zeta \in \mathrm{R}^n$, where $0<\alpha\le\beta<\infty$ are constants. Without saying it so every time we always assume that $1<p<\infty$. The very weak solution of equation (1.2) is understood in the sense of distributions. That is, *if u is a function in the Sobolev class $W^{1,r}_{loc}(R^n)$ with* $\max\{1, p-1\} \le r < p$ *and satisfies the integral relation*

$$\int \langle \mathcal{A}(x, g+\nabla u), \nabla\psi\rangle = \int f\psi \tag{1.3}$$

for all $\varphi \in C_0^\infty(R^n)$, then u is called a very weak solution of equation (1.2).

The papers [17] and [18] were the first to study the so-called *very weak* solutions. We explicitly remark that for very weak solutions, the usual truncation of the solution as a test function fails, see [17].

Let

$$\mathcal{D}^p(\mathrm{R}^n) = \{\phi : \phi \in L^1_{loc}(\mathrm{R}^n), \bigtriangledown\phi \in L^p(\mathrm{R}^n,\mathrm{R}^n), 1<p<\infty\}\,,$$

such gradient fields form a closed subspace, denoted by $\bigtriangledown\mathcal{D}^p(\mathrm{R}^n) \subset L^p(\mathrm{R}^n,\mathrm{R}^n)$. The truncation for a measurable function f is defined as

$$[f] = \begin{cases} |f|, & \text{if } |f| \le 1\,, \\ |f|^{-1}, & \text{if } |f| > 1\,. \end{cases}$$

The main result of this paper is the following theorem.

Theorem. *There exist exponents* p_1, p_2 *verifying* $\max\{1, p-1\} < p_1 < p < p_2$, *which have the following properties. For* $p_1 \le r \le p \le s \le p_2$ *and* $g \in L^s, f \in L^t$, *where*

$$t = \begin{cases} \frac{ns}{n(p-1)+s}, & \text{if} \quad p < n\,, \\ \frac{s}{p-1}, & \text{if} \quad p \ge n\,, \end{cases}$$

we have:

(a) equation(1.2) has unique solution of class $\mathcal{D}^s(\mathrm{R}^n)$ *and moreover*

$$\int |\nabla u|^s \, dx \le C \left(\|g\|_s^s + \|f\|_t^{s/(p-1)} \right) \tag{1.4}$$

with $C = C(p_1, p_2) > 0$.

(b) if $u \in \mathcal{D}^r(\mathrm{R}^n)$ *solve (1.2), then* $u \in \mathcal{D}^s(\mathrm{R}^n)$ *as well, and verifies (1.4).*

In order to prove the main theorem, we will need some preliminary results. We first recall that the usual Iwaniec-Hodge decomposition is for $u \in \mathcal{D}^p(\mathrm{R}^n), r = p - \varepsilon$, we can decompose $[\bigtriangledown u]^\varepsilon \bigtriangledown u$ as

$$[\bigtriangledown u]^\varepsilon \bigtriangledown u = \bigtriangledown \phi + H\,,$$

where $\mathrm{div} H = 0$ and

$$\int |H|| \bigtriangledown \phi|^{p-1} dx \le C_p \varepsilon \int [\bigtriangledown u]^\varepsilon | \bigtriangledown u|^p dx\,. \tag{1.5}$$

We also have the Sobolev inequality: *Let* $\Omega \subset R^n$ *be a cube and* $\phi \in W^{1,q}(\Omega)$ *with* $1 \le q < n$. *Then* $u \in L^{\frac{nq}{n-q}}$ *and we have the estimate*

$$\|\phi\|_{\frac{nq}{n-q}} \le \frac{nq-q}{n-q} \| \bigtriangledown \phi\|_q + |\Omega|^{-\frac{1}{n}} \|\phi\|_q\,. \tag{1.6}$$

It is not hard to see from the result above, that for an arbitrary domain Ω, if $u \in W_0^{1,p}(\Omega)$, then

$$\|\phi\|_{\frac{nq}{n-q}} \le \frac{nq-q}{n-q} \| \bigtriangledown \phi\|_q \tag{1.7}$$

Indeed, we may choose the function ϕ as an element of $W^{1,n}(\mathrm{R}^n)$ which is equal to 0 outside Ω. Then we apply (1.6) to arbitrarily large cubes in place of Ω, and find (1.7) as the limiting case. If, moreover, Ω has finite measure, then an application of Hölder's inequality gives the *Poincaré estimate*

$$\|u\|_p \le p|\Omega|^{1/n} \| \bigtriangledown u\|_p \tag{1.8}$$

for every $u \in W_0^{1,p}(\Omega)$ with $1 \le p < \infty$.

2. Proof of the Main Theorem

We shall prove the statement of the theorem with $r = p - \epsilon, s = p + \epsilon$, for $\epsilon \in (0,1)$ sufficiently small. We begin with a priori estimate which proves part (b) of the theorem.

Lemma 1. *If u solves (1.2), and $[\nabla u]^\epsilon \mid \nabla u \mid^p$ is integrable, then actually $\nabla u \in L^s$, and (1.4) holds.*

The proof of Lemma 1 is similar to the proof of [3, Proposition 4.1]. For the completion of this paper, we proof Lemma 1 as follows.

Sketch of the proof. We need only to replace

$$\int |h|| \bigtriangledown \phi| dx$$

by

$$\int |f||\phi| dx$$

in [3, Proposition 4.1]. If $p < n$, then $s = p + \varepsilon < n$ for ε sufficiently small. By using Hölder's inequality and (1.7), we obtain

$$\begin{aligned}\int |f||\phi| dx &\leq \|f\|_{\frac{ns}{n(p-1)+s}} \|\phi\|_{\frac{ns}{ns-n(p-1)-s}} \\ &\leq C\|f\|_{\frac{ns}{n(p-1)+s}} \| \bigtriangledown \phi\|_{\frac{s}{s-p+1}} \leq C\|f\|_{\frac{ns}{n(p-1)+s}} \left(\int [\bigtriangledown u]^\varepsilon | \bigtriangledown u|^p \right)^{(s-p+1)/s} .\end{aligned}$$

The last inequality being a consequence of the stability result (1.5) of the Iwaniec-Hodge decomposition and the inequality $([\bigtriangledown u]^\varepsilon |\bigtriangledown u|)^{s/(s-p+1)} \leq [\bigtriangledown u]^\varepsilon |\bigtriangledown u|^p$, see [3].

If $p \geq n$, then $s = p + \varepsilon > n$. By using Hölder's inequality and (1.8), we obtain

$$\begin{aligned}\int |f||\phi| dx &\leq \|f\|_{\frac{s}{p-1}} \|\phi\|_{\frac{s}{s-p+1}} \\ &\leq C\|f\|_{\frac{s}{p-1}} \| \bigtriangledown \phi\|_{\frac{s}{s-p+1}} \leq C\|f\|_{\frac{s}{p-1}} \left(\int [\bigtriangledown u]^\varepsilon | \bigtriangledown u|^p \right)^{(s-p+1)/s} .\end{aligned}$$

The other arguments are similar to the proof of Proposition 4.1 in [3], we only need to use (1.5) obtained from the results of Riesz transforms and interpolation. We omit the details. □

Having proved part (b) of Theorem and the estimate (1.4) of part (a), what remains is to prove the uniqueness result of part (a) of Theorem. It is clear if we assume $g \in L^p \bigcap L^s, f \in L^q \bigcap L^t$. In this case equation (1.2) has unique solution $u \in D^p$, which, by the above proposition, is in D^s and verifies (1.4). Next, we drop the assumption $g \in L^p, f \in L^q$ and prove existence by an approximation argument, which is based on the following lemma.

Lemma 2. *Let $u_1, u_2 \in W^{1,p}_{loc}$ be solutions to the nonhomogeneous $\mathcal{A}$-harmonic equations*

$$\mathrm{div}\mathcal{A}(x, g_j + \nabla u_j) = -f_j, j = 1, 2\,, \tag{2.1}$$

where $g_j \in L^p_{loc}, f_j \in L^q_{loc}, q = \frac{p}{p-1}$. Then for every nonnegative test function $\varphi \in C_0^\infty$ verifying $\varphi \cdot (g_1 - g_2) \equiv 0$ and $\varphi \cdot (f_1 - f_2) \equiv 0$, we have

$$\begin{aligned} \| \varphi(\nabla u_1 - \nabla u_2) \|_p \leq C \| \nabla\varphi(u_1 - u_2) \|_p^\alpha \\ \times (\| \varphi(g_1 + \nabla u_1) \|_p + \| \varphi(g_2 + \nabla u_2) \|)_p)^{1-\alpha}\,, \end{aligned} \tag{2.2}$$

with $C = C(p, \frac{\beta}{\alpha}) > 0$ and $\alpha \in (0,1)$ depending only on p.

Proof. Since $u_1, u_2 \in W^{1,p}_{loc}$ be solutions to (2.1) in the distributional sense, we obviously have

$$\int \langle \mathcal{A}(x, g_1 + \nabla u_1) - \mathcal{A}(x, g_2 + \nabla u_2), \nabla\psi \rangle \mathrm{d}x = \int (f_1 - f_2) \cdot \psi \mathrm{d}x$$

for any test function $\psi \in W_0^{1,p}$. Take $\psi = \varphi^p \cdot (u_1 - u_2) \in W_0^{1,p}$ in the above equality and since

$$\nabla\psi = \nabla(\varphi^p(u_1 - u_2)) = \varphi^p \nabla(u_1 - u_2) + p\varphi^{p-1}\nabla\varphi(u_1 - u_2)\,,$$

we then obtain

$$\begin{aligned} \int \langle \mathcal{A}(x, g_1 + \nabla u_1) - \mathcal{A}(x, g_2 + \nabla u_2), \varphi^p \nabla(u_1 - u_2) + p\varphi^{p-1}\nabla\varphi(u_1 - u_2)\rangle \mathrm{d}x \\ = \int (f_1 - f_2)\varphi^p(u_1 - u_2)\mathrm{d}x\,. \end{aligned}$$

From the assumption that $\varphi \cdot (f_1 - f_2) \equiv 0$ we conclude that the right-hand side of the above equality equals to zero. Therefore,

$$I_1 = \int \langle \mathcal{A}(x, g_1 + \nabla u_1) - \mathcal{A}(x, g_2 + \nabla u_2), \varphi^p \nabla(u_1 - u_2)\rangle \mathrm{d}x$$

$$= -\int \langle \mathcal{A}(x, g_1 + \nabla u_1) - \mathcal{A}(x, g_2 + \nabla u_2), p\varphi^{p-1}\nabla\varphi(u_1 - u_2)\rangle \mathrm{d}x$$
$$= I_2. \quad (2.3)$$

If $p > 2$, then the left-hand side of (2.3) can be estimated by the monotonicity assumption of $\mathcal{A}$ and the assumption $\phi(g_1 - g_2) \equiv 0$. That is,

$$\begin{aligned}
I_1 &= \int \langle \mathcal{A}(x, g_1 + \nabla u_1) - \mathcal{A}(x, g_2 + \nabla u_2), \varphi^p \nabla(u_1 - u_2)\rangle dx \\
&= \int \langle \mathcal{A}(x, g_1 + \nabla u_1) - \mathcal{A}(x, g_2 + \nabla u_2), \varphi^p (\nabla u_1 + g_1 - \nabla u_2 - g_2)\rangle dx \\
&\geq \alpha \int \varphi |g_1 + \bigtriangledown u_1 - g_2 - \bigtriangledown u_2|^2 (|g_1 + \bigtriangledown u_1| + |g_2 + \bigtriangledown u_2|)^{p-2} dx \\
&\geq \alpha \int \varphi^p |g_1 + \nabla u_1 - g_2 - \nabla u_2|^p dx \\
&\geq \alpha \int \varphi^p (2^{1-p}|\nabla u_1 - \nabla u_2|^p - |g_1 - g_2|^p) dx \\
&= 2^{1-p}\alpha \int \varphi^p |\nabla u_1 - \nabla u_2|^p dx\,. \quad (2.4)
\end{aligned}$$

By the Lipschitz type condition of $\mathcal{A}$ and Hölder's inequality, the right-hand side of (2.3) can be estimated as follows

$$\begin{aligned}
|I_2| &\leq \\
&\beta \int p\varphi^{p-1} |g_1 + \nabla u_1 - g_2 - \nabla u_2| (|g_1 + \nabla u_1| + |g_2 + \nabla u_2|)^{p-2} |\nabla\varphi| |u_1 - u_2| \mathrm{d}x \\
&\leq p\beta \int \varphi^{p-1} |\nabla u_1 - \nabla u_2| (|g_1 + \nabla u_1| + |g_2 + \nabla u_2|)^{p-2} |\nabla\varphi| |u_1 - u_2| \mathrm{d}x \\
&\leq p\beta \left(\int (\varphi |g_1 + \nabla u_1| + \varphi |g_2 + \nabla u_2|)^p \mathrm{d}x \right)^{(p-2)/p} \\
&\times \left(\int \varphi^p |\nabla u_1 - \nabla u_2|^p \mathrm{d}x \right)^{1/p} \times \left(\int |\nabla\varphi|^p |u_1 - u_2|^p \mathrm{d}x \right)^{1/p}. \quad (2.5)
\end{aligned}$$

Combining (2.3), (2.4) and (2.5) together, we find that

$$\left(\int \varphi^p |\nabla u_1 - \nabla u_2|^p \mathrm{d}x \right)^{(p-1)/p} \leq 2^{p-1} \frac{p\beta}{\alpha} \left(\int |\nabla\varphi|^p |u_1 - u_2|^p \mathrm{d}x \right)^{1/p}$$
$$\times \left(\int (\varphi |g_1 + \nabla u_1| + \varphi |g_2 + \nabla u_2|)^p \mathrm{d}x \right)^{(p-2)/p}$$

$$\leq 2^{p-1}\frac{p\beta}{\alpha}\|\nabla\varphi(u_1-u_2)\|_p(\|\varphi(g_1+\nabla u_1)\|_p+\|\varphi(g_2+\nabla u_2)\|_p)^{p-2}\,.$$

That is,

$$\|\varphi(\nabla u_1-\nabla u_2)\|_p\leq 2\left(\frac{p\beta}{\alpha}\right)^{1/(p-1)}\|\nabla\varphi(u_1-u_2)\|_p^{1/(p-1)}$$
$$\times\left(\|\varphi(g_1+\nabla u_1)\|_p+\|\varphi(g_2+\nabla u_2)\|_p\right)^{(p-2)/(p-1)}\,.$$

This inequality is equaivalent to (2.2), where $C=2\left(\frac{p\beta}{\alpha}\right)^{1/(p-1)}$ and $\alpha=\frac{1}{p-1}$.

If $1<p\leq 2$, then by (2.3) we have

$$\alpha\int\varphi^p|g_1+\triangledown u_1-g_2-\triangledown u_2|^2(|g_1+\triangledown u_1|+|g_2+\triangledown u_2|)^{p-2}dx\leq p\beta$$
$$\times\int[\varphi|g_1+\triangledown u_1-g_2-\triangledown u_2|][\varphi(|g_1+\triangledown u_1|+|g_2+\triangledown u_2|)]^{p-2}|\triangledown\phi(u_1-u_2)|dx$$
$$\leq p\beta\int[\varphi|g_1+\triangledown u_1-g_2-\triangledown u_2|]^{p-1}|\triangledown\phi(u_1-u_2)|dx\,. \quad (2.6)$$

The right-hand side of (2.6) can be estimated as follows

$$p\beta\int[\varphi|g_1+\triangledown u_1-g_2-\triangledown u_2|]^{p-1}|\triangledown\phi(u_1-u_2)|dx$$
$$\leq p\beta\|\varphi|g_1+\triangledown u_1-g_2-\triangledown u_2|\|_p^{p-1}\|\triangledown\phi(u_1-u_2)\|_p$$
$$\leq p\beta\left(\|\varphi(g_1+\triangledown u_1)\|_p+\|\varphi(g_2+\triangledown u_2)\|_p\right)^{p-1}\|\triangledown\phi(u_1-u_2)\|_p\,. \quad (2.7)$$

Therefore,

$$\int\varphi^p|\triangledown u_1-\triangledown u_2|^pdx=\int(\varphi|\triangledown u_1-\triangledown u_2|)^p$$
$$\times\left(\varphi(|g_1+\triangledown u_1|+|g_2+\triangledown u_2|)\right)^{\frac{p(p-2)}{2}}\left(\varphi(|g_1+\triangledown u_1|+|g_2+\triangledown u_2|)\right)^{\frac{p(2-p)}{2}}dx$$
$$\leq\left(\int(\varphi|\triangledown u_1-\triangledown u_2|)^2(\varphi(|g_1+\triangledown u_1|+|g_2+\triangledown u_2|))^{p-2}dx\right)^{\frac{p}{2}}$$
$$\times\left(\int(\varphi(|\triangledown u_1+g_1|+|\triangledown u_2+g_2|))^pdx\right)^{\frac{2-p}{2}}$$
$$\leq\left(\int(\varphi|g_1+\triangledown u_1-g_2-\triangledown u_2|)^2(\varphi(|g_1+\triangledown u_1|+|g_2+\triangledown u_2|))^{p-2}dx\right)^{\frac{p}{2}}$$

$$\times \left(\int (\varphi(| \bigtriangledown u_1 + g_1| + | \bigtriangledown u_2 + g_2|))^p dx \right)^{\frac{2-p}{2}} .$$

Using (2.6) and (2.7), we have

$$\begin{aligned}
&\int \varphi^p | \bigtriangledown u_1 - \bigtriangledown u_2|^p dx \\
&\quad \le \left(\frac{p\beta}{\alpha} (\|\varphi(g_1 + \bigtriangledown u_1)\|_p + \|\varphi(g_2 + \bigtriangledown u_2)\|_p)^{p-1} \| \bigtriangledown \phi(u_1 - u_2)\|_p \right)^{\frac{p}{2}} \\
&\qquad\qquad \times \left(\int (\varphi(|g_1 + \bigtriangledown u_1| + |g_2 + \bigtriangledown u_2|))^p dx \right)^{\frac{2-p}{2}} \\
&\quad \le \left(\frac{p\beta}{\alpha} \right)^{\frac{p}{2}} (\|\varphi(g_1 + \bigtriangledown u_1)\|_p + \|\varphi(g_2 + \bigtriangledown u_2)\|_p)^{\frac{p}{2}} \| \bigtriangledown \phi(u_1 - u_2)\|_p^{\frac{p}{2}} .
\end{aligned}$$

This is just (2.2) with $\alpha = \frac{1}{2}$. So, in both cases we have derived the desired inequality (2.2), completing the proof of Lemma 2.

Having proved Lemma 2, we can easily prove the uniqueness statement of the Theorem. The proof is similar to that of [CGI], we omit the details.

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