

STOCHASTIC VOLTERRA CONVOLUTION  
WITH LÉVY PROCESS

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**Abstract:** In the paper we study stochastic convolution appearing in Volterra equation driven by so called Lévy process. By Lévy process we mean a process with homogeneous independent increments, continuous in probability and cadlag.

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**Key Words:** stochastic Volterra equation, Lévy process, stochastic convolution

1. Introduction

Let  $H$  be a real separable Hilbert space with an inner product  $\langle \cdot, \cdot \rangle_H$  and a norm  $|\cdot|_H$ . In the paper we consider a stochastic version of linear, scalar type Volterra equation in  $H$  of the form

$$u(t) = \int_0^t a(t - \tau) Au(\tau) d\tau + x + g(t), \quad t \geq 0, \quad (1)$$

where  $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ ,  $A$  is an unbounded linear operator in  $H$  with a dense domain  $\mathcal{D}(A)$ ,  $g$  is an  $H$ -valued mapping and  $x \in H$ .

The linear integral equation (1) is a subject of many papers connected with applications in different fields. Among others, the equation (1) may be applied to several problems arising in mathematical physics. For instance, theory of viscoelasticity provides numerous problems leading to the Volterra equation of

the form (1) (see [9], for survey).

We assume that the equation (1) is *well-posed* and denote by  $\mathcal{R}(t), t \geq 0$ , the family of resolvent operators corresponding to (1). Operators  $\mathcal{R}(t)$  are linear for each  $t \geq 0$ , uniformly bounded on compact intervals,  $\mathcal{R}(0)x = x$  holds on  $\mathcal{D}(A)$ , and  $\mathcal{R}(t)x$  is continuous on  $\mathbb{R}_+$  for each  $x \in \mathcal{D}(A)$ . Additionally, the following *resolvent equation* holds

$$\mathcal{R}(t)x = x + \int_0^t a(t-\tau)A\mathcal{R}(\tau)x d\tau$$

for all  $x \in \mathcal{D}(A)$ ,  $t \geq 0$ . For more details concerning well-posedness and resolvent operators we refer again to the monograph [9].

In the paper we study equation (1) with an external force  $g(t) = Z(t), t \geq 0$ , where  $Z$  is a Lévy process defined on a stochastic basis  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ . This way the traditional Gaussian framework, when a Wiener process is the external noise, is extended. Our considerations are motivated by the growing interest in Lévy processes in applications, when the empirical observations simply cannot be explained by means of the Gaussian distribution.

So, we arrive at the equation

$$X(t) = \int_0^t a(t-\tau)AX(\tau) d\tau + X_0 + Z(t), \quad \text{where } X_0 \in H. \quad (2)$$

Stochastic equations of Volterra type driven by semimartingales have been of course already studied by some authors, for instance [7] or [11, 12]. But our paper treats the subject in a different spirit. We use the resolvent operators of the equation considered, then this way we try to extend the semigroup approach to the equation (2). For defining the stochastic convolution we do not use the general semimartingales technique but a simpler method.

We have assumed that  $A$  is a closed linear operator in  $H$  with the dense domain  $\mathcal{D}(A)$ . For such a class of operators exists the family  $\mathcal{R}(t), t \geq 0$ , of resolvent operators for (2) which fulfills some useful properties (see again [9], Chapter 1). Moreover, the family  $\mathcal{R}(t), t \geq 0$ , is a good enough class of operators to be integrands in stochastic integral with respect to a process with independent increments.

Now, we introduce definitions of solutions to (2), analogously like in previously considered stochastic cases.

**Definition 1.** An  $H$ -valued predictable process  $X(t), t \in [0, T]$ , is said to be a *weak solution* to (2), if  $P(\int_0^t |a(t-\tau)X(\tau)|_H d\tau < +\infty) = 1$  and if for all

$\xi \in \mathcal{D}(A^*)$  and all  $t \geq 0$  the following equation holds

$$\langle X(t), \xi \rangle_H = \langle X_0, \xi \rangle_H + \left\langle \int_0^t a(t-\tau)X(\tau)d\tau, A^*\xi \right\rangle_H + \langle Z(t), \xi \rangle_H, \quad P - \text{a.s.}$$

**Definition 2.** An  $H$ -valued predictable process  $X(t)$ ,  $t \in [0, T]$ , is said to be a *mild solution* to (2), if  $P(\int_0^T |X(\tau)|_H d\tau < +\infty) = 1$  and, for arbitrary  $t \in [0, T]$ ,

$$X(t) = \mathcal{R}(t)X_0 + \int_0^t \mathcal{R}(t-\tau) dZ(\tau), \quad (3)$$

where  $\mathcal{R}(t)$  is the resolvent for the equation (2).

The aim of the paper is to study process called *stochastic convolution*

$$Z_{\mathcal{R}}(t) := \int_0^t \mathcal{R}(t-\tau) dZ(\tau), \quad t \geq 0, \quad (4)$$

which is the crucial part of the solution (3).

In Section 2 we recall the rigorous definition of stochastic convolution with Lévy process. Then we adapt it to the Volterra equation (2) driven by Lévy process and use some properties of such integral. Next, in Section 3 we consider particular Volterra equations.

## 2. Stochastic Convolution

Assume that  $Z(t)$ ,  $t \geq 0$ , is an  $H$ -valued process with homogeneous independent increments (that is, Lévy process), defined on a fixed probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ , continuous in probability, cadlag and  $Z_0 = 0$ . It is of great importance in the study of linear and nonlinear stochastic Volterra equations driven by Lévy processes to establish first the basic properties of the process  $Z_{\mathcal{R}}(t) = \int_0^t \mathcal{R}(t-\tau) dZ(\tau)$ .

The stochastic convolution (4), where  $Z$  is a Lévy process, may be defined analogously like stochastic integral in the paper [2] that is, as a limit in probability of Stieltjes sums. This integral coincides with the integral defined by the semimartingales technique (see [6] or [8]), but approach used in [2] provides immediately some useful properties of the integral. The most important is that we obtain the explicit formula for the characteristic form of the convolution (4).

First we recall some facts concerning stochastic integral with respect to Lévy process used in the paper.

The class  $\mathcal{L}_{[u,w]}^2(H, G)$  of integrands is defined as follows:

$\Phi : [u, w] \longrightarrow L(H, G)$ , where  $u, w \in \mathbb{R}_+$  and  $H, G$  are Hilbert spaces, such that:

- 1) for any  $h \in H$ ,  $\Phi h : [u, w] \longrightarrow G$  is measurable,
- 2)  $\int_u^w \|\Phi(s)\|^2 ds < +\infty$ , where  $\|\cdot\|$  means the operator norm.

In the above definition  $L(H, G)$  denote the space of linear bounded operators acting from  $H$  into  $G$ .

**Theorem 1.** (see Theorem 1.3, [2]) *Let  $H$  and  $G$  be Hilbert spaces and a function  $\Phi$  belong to the class  $\mathcal{L}_{[u,w]}^2(H, G)$ . Assume that there exists a sequence  $\{\Phi_n\}$  of step functions that:*

- for any  $n \in N$ ,  $\Phi_n : [u, w] \rightarrow L(H, G)$  and for partition  $u = s_0 < s_1 < \dots < s_n = t$ ,  $\Phi_n(s) = \Phi_n^k$ , where  $s \in (s_k, s_{k+1}]$ ,  $k = 0, 1, \dots, n-1$ ;
- the following condition holds

$$\Phi(s)h = \lim_{n \rightarrow \infty} \Phi_n(s)h \quad \text{for any } h \in H \text{ for a.a. } s \in [u, w]; \quad (5)$$

- there exists a function  $g \in L^1([u, w])$  that

$$\sup_n \|\Phi_n(s)\|^2 \leq g(s) \quad \text{for a.a. } s \in [u, w]. \quad (6)$$

Then the sequence of random variables

$$J(\Phi_n) := \int_{(u,w]} \Phi_n(s) dZ_s = \sum_{k=0}^{n-1} \Phi_n^k (Z_{s_{k+1}} - Z_{s_k})$$

converges in probability and the limit does not depend on the choice of the sequence  $\{\Phi_n\}$ .

**Theorem 2.** (see Theorem 1.8, [2]) *Let the function  $\Phi$  and the sequence  $\{\Phi_n\}$  satisfy the assumptions of Theorem 1. Then the integral defined as*

$$\int_{(u,w]} \Phi(s) dZ_s := P - \lim_{n \rightarrow \infty} \int_{(u,w]} \Phi_n(s) dZ_s \quad (7)$$

is well-defined  $G$ -valued random variable which has infinitely divisible distribution.

**Remark 1.** The integral  $\int_u^\infty \Phi(s) dZ_s$  is defined as the limit in probability, as  $w \rightarrow \infty$ , of the integrals  $\int_{(u,w]} \Phi(s) dZ_s$ . The integrals  $\int_{-\infty}^t \Phi(s) dZ_s$  and  $\int_{-\infty}^{+\infty} \Phi(s) dZ_s$  are defined analogously.

As we have already written, the aim of the paper is to study the stochastic convolution (4), where  $Z(t)$ ,  $t \geq 0$ , is a Lévy process and  $\mathcal{R}(t)$ ,  $t \geq 0$ , are the resolvent operators to the Volterra equation (2). Basing on properties of resolvent operators we can see that the operators  $\mathcal{R}(t)$ ,  $t \in [u, w]$ , belong to the class  $\mathcal{L}^2_{[u,w]}(H, H)$ . Actually,  $\mathcal{R}(t)$  are linear and bounded for each  $t \geq 0$  and  $\mathcal{R}(t)x$  is continuous on  $\mathbb{R}_+$  for each  $x$  belonging to the domain  $\mathcal{D}(A)$  of the operator  $A$ . Moreover, the function  $\mathcal{R}(\cdot)x$ ,  $x \in \mathcal{D}(A)$ , is measurable. Let  $\mathcal{R}_n(t)$ ,  $t \in [u, w]$ , be step functions defined as follows

$$\mathcal{R}_n := \sum_{i=1}^n \mathcal{R}(s_i) \chi_{[t_{i-1}, t_i]}, \tag{8}$$

where  $t_0 = u$ ,  $t_0 < t_1 < \dots < t_{i-1} < t_i < \dots < t_n = w$ , and  $s_i$  is a point from  $[t_{i-1}, t_i]$ .

Let us notice that the functions  $\mathcal{R}_n$  defined by (8) on the interval  $[u, w]$ , satisfy conditions (5) and (6). Indeed, the interval  $[u, w]$  is a compact set in  $\mathbb{R}$  and the operator  $\mathcal{R}(t)$ ,  $t \in [u, w]$  is continuous with respect to  $t$ . Then the sequence  $(\mathcal{R}_n)$ ,  $n \in N$ , of step functions (8) is uniformly convergent to the function  $\mathcal{R}$ . Additionally  $(\mathcal{R}_n)$ ,  $n \in N$ , are bounded.

So, we may define the stochastic convolution (4) like the stochastic integral (7), that is, like the limit in probability of integrals of step functions. Hence, the following theorem comes directly from Theorem 2.

**Theorem 3.** *Let  $\mathcal{R}(t)$ ,  $t \geq 0$ , be the family of resolvent operators of the Volterra equation (2) and  $Z(t)$ ,  $t \geq 0$ , be a Lévy process. Then the integral*

$$\int_0^t \mathcal{R}(t - \tau) dZ(\tau) := P - \lim_{n \rightarrow +\infty} \int_0^t \mathcal{R}_n(t - \tau) dZ(\tau)$$

*is well-defined  $H$ -valued random variable which has infinitely divisible distribution.*

Let us recall that the process  $Z(t)$ ,  $t \geq 0$ , as the process with independent increments, has the following representation (see for instance [1, 3, 5] or [10])

$$Z_t = at + W_t + \Delta_t,$$

where  $a \in H$ ,  $(W_t), t \geq 0$ , is an  $H$ -valued Wiener process and  $(\Delta_t), t \geq 0$ , is a jump process independent of  $(W_t)$ . This decomposition is clearly unique. Moreover, for any  $t \geq 0$ , the random variable  $Z_t$  has Lévy characterization  $[ta, t\Theta, tM]$ , where  $\Theta$  is the covariance operator of  $W_1$  and  $M$  is the Lévy spectral measure of  $\Delta_1$ . This is a consequence of the fact that any infinitely divisible

probability measure can be viewed as the distribution of a Lévy process evaluated at time 1 and vice versa. Particularly, the famous Lévy-Khintchine formula determines the class of characteristic functions corresponding to infinitely divisible laws.

Now, we are ready to characterize the convolution (4) as follows.

**Theorem 4.** *Let Lévy process  $Z(t)$ ,  $t \geq 0$ , be such that every random variable  $Z_t$  has Lévy characterization  $[ta, t\Theta, tM]$ . Then the stochastic convolution  $Z_{\mathcal{R}}(t) = \int_0^t \mathcal{R}(t-\tau) dZ(\tau)$ ,  $t \geq 0$ , where  $\mathcal{R}(t)$  are resolvent operators to the Volterra equation (2), has the following Lévy characterization  $[\alpha, Q, \mathcal{M}]$ :*

$$\begin{aligned} \alpha &= \int_0^t \mathcal{R}(t-\tau) a d\tau \\ &\quad + \int_0^t \int_H \mathcal{R}(t-\tau) x [\mathbf{1}_{\{|\mathcal{R}(t-\tau)x| < 1\}} - \mathbf{1}_{\{|x| < 1\}}] M(dx) d\tau ; \\ Q &= \int_0^t \mathcal{R}(t-\tau) \Theta \mathcal{R}^*(t-\tau) d\tau ; \\ \mathcal{M} &= \int_0^t M(\Phi^{-1}(t-\tau) dx) d\tau . \end{aligned} \tag{9}$$

*Proof.* By Theorem 3, the stochastic convolution given by the formula  $Z_{\mathcal{R}}(t) = \int_0^t \mathcal{R}(t-\tau) dZ(\tau)$ ,  $t \geq 0$ , has infinitely divisible distribution. In order to provide the Lévy characterization of  $Z_{\mathcal{R}}(t)$  it is enough to write the characteristic functional of the law of  $Z_{\mathcal{R}}(t)$  and next use the Lévy-Khintchine formula for the corresponding characteristic exponent. The proof of the theorem is analogous to the proofs of Lemma 1.5 and Theorem 1.8 from the paper [2]. In our case, the characteristic functional is  $f(y) := \exp \int_0^t \phi(\mathcal{R}(t-s)y) ds$ , for  $y \in H$ , where  $\phi(w) = \log \mathbb{E}(\exp i\langle w, Z_1 \rangle)$ ,  $w \in H$ .  $\square$

**Corollary 1.** *The stochastic convolution (4) is stochastically continuous and then has a predictable version.*

**Theorem 5.** *Assume that the operators  $\mathcal{R}(t)$ ,  $t \geq 0$ , are as above. Then the stochastic Volterra equation (2) has exactly one mild solution.*

**Comment.** Theorem 5 comes from uniqueness of the resolvent  $\mathcal{R}(t)$ ,  $t \geq 0$ , for deterministic Volterra equation (1) and the existence of mild solution to (1).

**Theorem 6.** *Assume that the operators  $A, \mathcal{R}(t)$  and the process  $Z(t)$ ,  $t \geq 0$ , are like above and the function  $a \in W_{\text{loc}}^{1,1}(\mathbb{R}^+)$ . Let  $X$  be an  $H$ -valued predictable process with integrable trajectories. If for any  $t \in [0, T]$  and  $\xi \in$*

$\mathcal{D}(A^*)$  the equality

$$\langle X(t), \xi \rangle_H = \int_0^t \langle a(t-\tau)X(\tau), A^*\xi \rangle_H d\tau + \int_0^t \langle \xi, dZ(\tau) \rangle_H \quad (10)$$

holds, then

$$X(\cdot) = Z_R(\cdot). \quad (11)$$

*Proof.* The idea of the proof is the following. First, we prove that if (10) is satisfied, then for any  $\tilde{\xi} \in C^1([0, T], \mathcal{D}(A^*))$  and  $t \in [0, T]$ , the following equality holds

$$\begin{aligned} \langle X(t), \tilde{\xi}(t) \rangle_H &= \int_0^t \langle (\dot{a} \star X)(\tau) + a(0)X(\tau), A^*\tilde{\xi} \rangle_H d\tau + \int_0^t \langle \tilde{\xi}(\tau), dZ(\tau) \rangle_H \\ &\quad + \int_0^t \langle X(\tau), \tilde{\xi}(\tau) \rangle_H, \quad P - a.s. \quad (12) \end{aligned}$$

Next, we take  $\tilde{\xi}(\tau) := \mathcal{R}^*(t-\tau)\xi$  for  $\tau \in [0, t]$ ,  $\xi \in \mathcal{D}(A^*)$  and rewrite (12).

Then, using properties of resolvent operators, particularly the resolvent equation, we obtain thesis (11).  $\square$

**Comment.** The Theorem 6 above says that a weak solution to (2) is a mild solution to (2).

**Theorem 7.** *The stochastic convolution  $Z_R(t) = \int_0^t \mathcal{R}(t-\tau)dZ(\tau)$ ,  $t \in [0, T]$ , fulfills the equation (10).*

*Proof.* Let us notice that the process  $Z_R$  has integrable trajectories (the set of „discontinuity” is at most countable).

For any  $\xi \in \mathcal{D}(A^*)$  we may write

$$\begin{aligned}
& \int_0^t \langle a(t-\tau)Z_R(\tau), A^*\xi \rangle_H d\tau \\
(\text{from (4)}) &= \int_0^t \langle a(t-\tau) \int_0^\tau \mathcal{R}(\tau-\sigma) dZ(\sigma), A^*\xi \rangle_H \\
(\text{from Dirichlet Formula and Stochastic Fubini Theorem}) & \\
&= \int_0^t \left\langle \left[ \int_0^t a(t-\tau) \mathcal{R}(\tau-\sigma) d\tau \right] dZ(\sigma), A^*\xi \right\rangle_H \\
&= \left\langle \int_0^t \left[ \int_0^{t-\sigma} a(t-\sigma-z) \mathcal{R}(z) dz \right] dZ(\sigma), A^*\xi \right\rangle_H \\
&= \left\langle \int_0^t A[(a \star \mathcal{R})(t-\sigma)] dZ(\sigma), \xi \right\rangle_H \\
(\text{from resolvent equation}) & \\
&= \left\langle \int_0^t [\mathcal{R}(t-\sigma) - I] dZ(\sigma), \xi \right\rangle_H \\
&= \left\langle \int_0^t \mathcal{R}(t-\sigma) dZ(\sigma), \xi \right\rangle_H - \left\langle \int_0^t dZ(\sigma), \xi \right\rangle_H.
\end{aligned}$$

Hence, we obtained the following equation

$$\langle Z_R(t), \xi \rangle_H = \int_0^t \langle a(t-\tau)Z_R(\tau), A^*\xi \rangle_H d\tau + \int_0^t \langle \xi, dZ(\tau) \rangle_H$$

for any  $\xi \in \mathcal{D}(A^*)$ . □

**Corollary 2.** *Assume that the operator  $A$  is bounded. Then*

$$Z_R(t) = \int_0^t a(t-\tau)AZ_R(\tau)d\tau + \int_0^t dZ(\tau).$$

### 3. Particular Cases

Let us notice that till now we have not assumed that the resolvent operators  $\mathcal{R}(t)$ ,  $t \geq 0$ , to the Volterra equation (2) have bounded variation. The stochastic integral with respect to Lévy process and the stochastic convolution (4) have been defined and characterized (formula (9)) without this assumption. Hence,

if the stochastic Volterra equation (2) driven by Lévy process  $Z(t)$ ,  $t \geq 0$ , is well-posed, that is, the equation (2) has the resolvents  $\mathcal{R}(t)$ ,  $t \geq 0$ , then Theorem 4 for the convolution  $Z_R(t)$ ,  $t \geq 0$ , holds. Besides, there are some Volterra equations which admit resolvent operators with bounded variation. These equations may be treated in a different way. So, in this section we study such particular case of Volterra equations.

We will take the following common assumption.

**Assumption 1.** There exists the family  $\mathcal{R}(t)$ ,  $t \geq 0$ , of resolvent operators to the Volterra equation (2), the operators have bounded variation and  $Z(t)$ ,  $t \geq 0$ , is an  $H$ -valued Lévy process.

If the resolvent operators  $\mathcal{R}(t)$ ,  $t \geq 0$ , have bounded variation, we may use for stochastic convolution (4) the classical integration by parts. Then we may write

$$\int_{(a,b]} \mathcal{R}(t - \tau)dZ(\tau) = \mathcal{R}(t - b)Z(b) - \mathcal{R}(t - a)Z(a) - \int_{(a,b]} Z(\tau-)d\mathcal{R}(t - \tau), \quad (13)$$

where  $Z(t)$ ,  $t \geq 0$ , is a stochastic Lévy process and  $Z(\tau-) = \lim_{u \uparrow \tau} Z(u)$ .

The properties of the stochastic integral  $\int_{(a,b]} \mathcal{R}(t - \tau)dZ(\tau)$  may be obtained from the right hand side of (13). Among others, we can deduce the following results.

**Proposition 1.** Under the assumption (A) we have

$$\langle x^*, \int_{(a,t]} \mathcal{R}(t - \tau)dZ(\tau) \rangle = \int_{(a,t]} \mathcal{R}(t - \tau)d\langle x^*, Z(\tau) \rangle$$

for  $x^* \in H^*$ , where  $H^*$  is the dual space to  $H$ .

**Proposition 2.** If the assumption (A) holds, the function

$$t \longrightarrow \int_{(a,t]} \mathcal{R}(t - \tau)dZ(\tau)$$

is  $H$ -valued random variable with infinite divisible distribution.

This fact follows from (13) and the approximation by Riemann-Stieltjes sums, see, e.g. [4].

**Proposition 3.** *Let the assumption (A) be satisfied. Then*

$$\begin{aligned} \log \left[ \hat{\mathcal{L}} \left( \int_{(0,t]} \mathcal{R}(t-\tau) dZ(\tau) \right) (\lambda) \right] \\ = \int_{(0,t]} \log [\hat{\mathcal{L}}(Z(1))(\lambda \mathcal{R}(t-s))] ds, \end{aligned} \quad (14)$$

for  $\lambda \in H^*$ , where  $H^*$  is the dual space to  $H$  and  $\hat{\mathcal{L}}$  denotes the characteristic functional of the probability distribution of the appropriate random variable.

*Proof.* The above formula (14) comes from Lemma 1.1, [4] and the definition of the convolution (4).  $\square$

**Proposition 4.** *If the assumption (A) holds, then  $\int_{(0,t]} \mathcal{R}(t-\tau) dZ(\tau)$  converges in norm w.p. 1, as  $t \rightarrow +\infty$ , if and only if  $\int_{(0,t]} \mathcal{R}(t-\tau) dZ(\tau)$  converges in the distribution as  $t \rightarrow \infty$ .*

*Proof.* Proposition comes from Lemma 1.2, [4].  $\square$

We finish the paper by the examples of Volterra equations.

Let us consider the case when the function  $a$  is completely positive. This class of kernels is very important in the theory of Volterra equations and arises naturally in applications. If the function  $a \in L^1_{\text{loc}}(\mathbb{R}_+)$  is completely positive, then  $s(\cdot, \gamma)$ , the solution to the integral equation

$$s(t) + \gamma \int_0^t a(t-\tau) s(\tau) d\tau = 1, \quad t \geq 0, \quad (15)$$

is nonnegative and nonincreasing for any  $\gamma > 0$ . More precisely, under this condition  $s(t) \in [0, 1]$ . There is a relationship between the resolvent operator  $\mathcal{R}(t)$  to the equation (2) and the corresponding function  $s(t, \gamma)$  fulfilling (15). Namely, if  $-\gamma$  is an eigenvalue of  $A$  with eigenvector  $z \neq 0$ , then  $\mathcal{R}(t)z = s(t, \gamma)z$ ,  $t \geq 0$ .

Because every function monotonic on the interval  $[0, T]$  has bounded variation on  $[0, T]$ , it is enough to choose any function  $a(t)$ ,  $t \geq 0$ , which is completely positive.

Particularly, let  $H = L^2(0, 1)$  and  $Au = D^2u$  with  $\mathcal{D}(A) = H^2(0, 1) \cap H^1_0(0, 1)$ . The functions  $e_k(\xi) = \sqrt{2/\pi} \sin k\xi$ ,  $\xi \in [0, 1]$ ,  $k \in N$ , form an orthonormal sequence of eigenfunctions of the operator  $A$ , corresponding to the eigenvalues  $-\mu_k = -\pi^2 k^2$ ,  $k \in N$ . When we set additionally, that the function  $a(t) = e^{-t}$  for  $t \geq 0$ , then we obtain in this case

$$s(t, \mu) = (1 + \mu)^{-1} [1 + \mu e^{-(1+\mu)t}], \quad t, \mu > 0.$$

There exists the resolvent  $\mathcal{R}(t)$ ,  $t \geq 0$ , to the equation (2) in this case and is determined by  $\mathcal{R}(t) e_k = s(t, \mu_k) e_k$ ,  $k \in N$ . So, the function  $\mathcal{R}(t)$ ,  $t \geq 0$ , is monotonic in  $[0, 1]$ , because the function  $s(t, \mu)$  is. Hence,  $\mathcal{R}(t)$  has bounded variation on  $[0, 1]$ .

Other examples of Volterra equations with resolvent operators having bounded variation may be found in the monograph [9].

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