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WADA'S REPRESENTATION AND THE NATURAL MAP $B_n \to B_{2n}$

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Abstract: We consider the linear representation of the braid group on n strands by automorphisms of the free group F_n; the representation that is discovered by M. Wada. Using the special case of Cohen's map $B_n \to B_{2n}$ and composing it with Wada's embedding $B_{2n} \to \mathrm{Aut}(F_{2n})$, we get a linear representation of degree $2n$ which has a subrepresentation isomorphic to the well-known Burau representation.

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1. Introduction

We denote the braid group on n strands by B_n. There is a well-known representation (due to Artin) of the braid group in the group $\mathrm{Aut}(F_n)$ of automorphisms of the free group F_n. Let F_n be the group generated by $x_1, \ldots, x_n$. Then the automorphism corresponding to the braid generator σ_i takes x_i to $x_i x_{i+1} {x_i}^{-1}$; x_{i+1} to x_i, and fixes all other free generators. Applying the free differential calculus to elements of $\mathrm{Aut}(F_n)$ sometimes gives rise to linear representations of B_n and the Burau representation arises this way.

Another type of representation, introduced by F.R. Cohen, is the map $B_n \to B_{nk}$, which is defined on geometric braids by replacing each string with k strings.

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Using the standard Artin representation, we have proved that by composing Cohen's map $B_n \to B_{nk}$ and the embedding $B_{nk} \to \mathrm{Aut}\,(F_{nk})$, we get a linear representation whose composition factors are one copy of the Burau representation and $k-1$ copies of the standard representation, a representation investigated by I. Sysoeva. For more details, see [1].

More recently, Wada [5] has discovered another representation of the group B_n by automorphisms of F_n. It was shown in [4] that such a representation is faithful. We will use this embedding of B_n into $\mathrm{Aut}\,(F_n)$ to get a linear representation of degree $2n$. This will be done using the special case of Cohen's map, namely, $B_n \to B_{2n}$. We show that the composition factors of the obtained representation has a subrepresentation isomorphic to the Burau representation.

2. Notation

The braid group of n strings, B_n, is an abstract group which has a presentation with generators:

$$\sigma_1, \ldots, \sigma_{n-1}$$

and defined by the relations:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for} \quad i = 1, 2, ..., n-2\,,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if} \quad |i-j| \geq 2\,.$$

The generators $\sigma_1, \ldots, \sigma_{n-1}$ are called the standard generators of B_n. Let t be an indeterminate and $\mathbb{C}[t^{\pm 1}]$ represent the Laurent polynomial ring over complex numbers.

Definition 1. The *Burau representation* $\beta_n(t) : B_n \to GL_n(\mathbb{C}[t^{\pm 1}])$ is defined by

$$\beta_n(t)(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 1-t & t & 0 \\ & 1 & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_{n-i-1} \end{array}\right), \text{ for } i = 1, \ldots, n-1.$$

Definition 2. The *Wada representation* asserts that the automorphism corresponding to σ_i takes

$$x_i \to x_i {x_{i+1}}^{-1} x_i\ , \quad x_{i+1} \to x_i\ ,$$
$$x_j \to x_j \ \text{ for } \ j \neq i, i+1\ .$$

By applying the Magnus representation to the image of the braid group under Wada's representation, we determine the matrix representations: $B_n \to GL_n(\mathbb{C}[t^{\pm 1}])$. We now introduce Fox derivatives defined as follows.

Definition 3. (see [2], p. 104) Let F_n be a free group of rank n, with free basis $x_1, \ldots, x_n$. We define for $j = 1, 2, \ldots, n$ the *free derivatives* on the group $\mathbb{Z}F_n$ by:

(i) $\dfrac{\partial x_i}{\partial x_j} = \delta_{i,j}$.

(ii) $\dfrac{\partial x_i^{-1}}{\partial x_j} = -\delta_{i,j} x_i^{-1}$.

(iii) $\dfrac{\partial}{\partial x_j}(uv) = \dfrac{\partial u}{\partial x_j}\epsilon(v) + u\dfrac{\partial v}{\partial x_j} \quad u\,, v \in \mathbb{Z}F_n$.

Note that if $v \in F_n$, then $\epsilon(v) = 1$. Here $\delta_{i,j}$ is the Kronecker symbol.

3. Cohen Representation

Definition 4. (see [3], p. 208) The *Cohen representation* is the map $B_n \to B_{nk}$ defined as follows:

$$\begin{aligned}\sigma_i &\to 1 \times \sigma_i \\ &= (\sigma_{ki}\sigma_{ki+1}...\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}...\sigma_{ki+k-2})...(\sigma_{ki-k+1}\sigma_{ki-k+2}...\sigma_{ki})\,.\end{aligned}$$

Here $1 \times \sigma_i$ is the braid obtained by replacing each string of the geometric braid, σ_i, with k parallel strings. Cohen called $1 \times \sigma_i$ a tensor product.

Here, we take the special case $k = 2$. Our objective is to construct a linear representation of B_n of degree $2n$ in the following way: Consider the composition map: $B_n \to B_{2n} \to GL_{2n}(\mathbb{Z}[t^{\pm 1}])$, where the first map is Cohen representation and the second one is the Wada's representation. Next we find a set of free generators for the group F_{2n}. Then we let τ_i be the image of the braid generator σ_i under the Cohen map and determine the action of the automorphism corresponding to τ_i on this basis of F_{2n}. This action is determined using Wada's representation of braids as automorphisms of a free group (see Definition 2). After applying free differential calculus to this element of $\mathrm{Aut}\,(F_{2n})$, we get the linear representation of degree $2n$ as desired.

Given the generators of F_n, namely, $x_1, \ldots, x_n$, we choose a certain basis of elements y_i, each of which is a word in these x_i's. More precisely, we have

$$y_1 = x_1{x_2}^{-1},\ \ y_2 = x_3{x_4}^{-1}, \dots,\ y_i = x_{2i-1}{x_{2i}}^{-1}, \dots,\ y_n = x_{2n-1}{x_{2n}}^{-1},$$
$$y_{n+1} = x_1,\ y_{n+2} = x_3, \dots,\ y_{n+i} = x_{2i-1}, \dots,\ y_{2n} = x_{2n-1}.$$

Theorem 1. *For $1 \le i \le n-1$ and $1 \le j \le 2n$, the action of τ_i on the basis $\{ y_j \}$ of F_{2n} is given by*

(i) $y_i \to y_i y_{i+1} {y_i}^{-1}$.

(ii) $y_{i+1} \to y_i$.

(iii) $y_r \to y_r, \quad (r \ne i,\ \ i+1,\ \ n+i,\ \ n+i+1)\ ,\ 1 \le r \le 2n$.

(iv) $y_{n+i} \to y_i (y_{n+i+1} {y_{n+i}}^{-1} y_i y_{n+i})$.

(v) $y_{n+i+1} \to y_{n+i}$.

Proof. Notice that the action of σ_i's on the generators x_i 's is given by M. Wada (see Definition 2).

Proof of (i). We have that $\tau_i = (\sigma_{2i}\sigma_{2i+1})(\sigma_{2i-1}\sigma_{2i})$. Then

$$\begin{aligned}(x_{2i-1}{x_{2i}}^{-1})\sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i} &= (x_{2i-1}{x_{2i}}^{-1}x_{2i+1}{x_{2i}}^{-1})\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}\\ &= (x_{2i-1}{x_{2i}}^{-1}x_{2i+1}{x_{2i+2}}^{-1}x_{2i+1}{x_{2i}}^{-1})\sigma_{2i-1}\sigma_{2i}\\ &= (x_{2i-1}{x_{2i}}^{-1}x_{2i-1}{x_{2i-1}}^{-1}x_{2i+1}{x_{2i+2}}^{-1}x_{2i+1}{x_{2i-1}}^{-1})\sigma_{2i}\\ &= x_{2i-1}{x_{2i}}^{-1}x_{2i+1}{x_{2i+2}}^{-1}x_{2i}{x_{2i-1}}^{-1} = y_i y_{i+1} {y_i}^{-1}\,.\end{aligned}$$

Proof of (ii). $y_{i+1} = x_{2i+1}{x_{2i+2}}^{-1}$. Then

$$\begin{aligned}(x_{2i+1}{x_{2i+2}}^{-1})\sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i} &= (x_{2i}{x_{2i+2}}^{-1})\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}\\ &= (x_{2i}{x_{2i+1}}^{-1})\sigma_{2i-1}\sigma_{2i} = (x_{2i-1}{x_{2i+1}}^{-1})\sigma_{2i} = x_{2i-1}{x_{2i}}^{-1} = y_i\,.\end{aligned}$$

Proof of (iii). If $1 \le r \le n$, then $y_r = x_{2r-1}{x_{2r}}^{-1}, \quad r \ne i, i+1$. Since $1 \le 2r-1 < 2i-1$ or $2i+1 < 2r-1 \le 2n-1$, it follows that the largest possible value of $2r-1$ is $2i-3$ or the smallest possible value of $2r-1$ is $2i+3$ respectively. In either case, y_r is fixed under the action of τ_i.

If $n+1 \le r \le 2n$ then $r = n+s$, where $1 \le s \le n, \quad s \ne i, i+1$. Here, we have $y_{n+s} = x_{2s-1}$. It is then easy to see that $2s-1$ cannot be $2i-1,\ 2i,\ 2i+1,\ 2i+2$ which implies that $y_{n+s} \to y_{n+s}$.

Proof of (iv). Let $y_{n+i} = x_{2i-1}$. Then

$$\begin{aligned}(x_{2i-1})\sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i} &= (x_{2i-1})\sigma_{2i-1}\sigma_{2i} = (x_{2i-1}{x_{2i}}^{-1}x_{2i-1})\sigma_{2i}\\ &= x_{2i-1}{x_{2i}}^{-1}x_{2i+1}{x_{2i}}^{-1}x_{2i-1} = y_i(y_{n+i+1}{y_{n+i}}^{-1}y_i y_{n+i})\,.\end{aligned}$$

Proof of (v). $y_{n+i+1} = x_{2i+1}$. Then it is easy to see that $y_{n+i+1} \to y_{n+i}$. □

Let ϕ be a homomorphism from F_{2n} to $\mathbb{C}^*$ defined by $\phi(y_i) = t$, for $1 \leq i \leq 2n$. Let $D_i = \phi\dfrac{\partial}{\partial y_i}$. Then we define the Jacobian matrix as follows:

$$J(\tau_i) = \begin{pmatrix} D_1(\tau_i(y_1)) & \dots & D_{2n}(\tau_i(y_1)) \\ \vdots & & \vdots \\ D_1(\tau_i(y_{2n})) & \dots & D_{2n}(\tau_i(y_{2n})) \end{pmatrix}.$$

We now prove our main theorem.

Theorem 2. *The linear representation obtained by composing the Cohen representation with the Wada's representation has a subrepresentation isomorphic to the (unreduced) Burau representation. That is, the image of* σ_i *is*

$$\left(\begin{array}{c|c} \beta_n(t)(\sigma_i) & 0 \\ \hline P_n & Q_n \end{array}\right),$$

where P_n *is an* $n \times n$ *matrix whose* (i,i)*-th entry is* $1+t$ *and all other entries are zeros. Also* Q_n *is the* $n \times n$ *matrix which differs from the identity matrix only by a* 2×2 *block with the top left corner in the* (i,i)*-th place. This block is* $\begin{pmatrix} -t+t^2 & t \\ 1 & 0 \end{pmatrix}$.

Proof. Take the subrepresentation as the one specified by the basis, $\{y_1, y_2, \dots, y_n\}$. By Theorem 1 (parts(i), (ii) and (iii)), and by applying Fox derivatives, we can easily get that

$$D_i(\tau_i(y_i)) = 1 - t \quad \text{and} \quad D_{i+1}(\tau_i(y_i)) = t\,.$$

Also,

$$D_i(\tau_i(y_{i+1})) = 1 \quad \text{and} \quad D_{i+1}(\tau_i(y_{i+1})) = 0,$$
$$D_k(\tau_i(y_j)) = \delta_{k,j}, \quad \text{if} \quad 1 \leq j \leq n, (j \neq i, i+1).$$

On the other hand, we have

$$D_{n+i}(\tau_i(y_{n+i})) = -t + t^2, \quad \text{and} \quad D_{n+i+1}(\tau_i(y_{n+i})) = t,$$
$$D_{n+i}(\tau_i(y_{n+i+1})) = 1, \quad \text{and} \quad D_{n+i+1}(\tau_i(y_{n+i+1})) = 0.$$ □

In a similar manner, we might try to generalize our work. That is, for any nonzero positive integer, k, we compose Cohen's map $B_n \to B_{nk}$ with Wada's representation to get a linear representation of degree nk. But here, we need to introduce a general basis for F_{nk} that implies our choice in the case $k = 2$.

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