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NEW WAY FOR A TWO-PARAMETER CANONICAL FORM OF SEXTIC EQUATIONS AND ITS SOLVABLE CASES

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Abstract: A new way for a two-parameter canonical form of sextic equations is given, through a Tschirnhausian transform by multiple steps, supplemented with some solvable cases at most in terms of hypergeometric functions.

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1. Introduction

Actually the general sextic equation $x^6 + \sum_{n=1}^{6} a_n x^{6-n} = 0$ can be solved in terms of Kampé de Fériet functions, and it can be reduced to a three-parameter

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resolvent of the form $y^6 + ay^4 + by^2 + cy + c = 0$ by Joubert [3]. Such a type of the sextic equation was treated by Felix Klein and a part was given in [4]. Other contributions to treatment on sextic equations were given e.g. by Cole [2] and Coble [1]. In this paper, a new way leading to a two-parameter canonical form of sextic equations is given.

2. Analysis

2.1. General

Let the original given sextic equation be expressed as

$$x^6 + a^*x^5 + b^*x^4 + c^*x^3 + d^*x^2 + e^*x + f^* = 0. \tag{1}$$

Here the quantity $5a^{*3} - 18a^*b^* + 27c^*$ corresponding to the polynomial of the left hand side of equation (1) is invariant with respect to the parallel shift $x \to x$+constant.

2.2. In Case of $5a^{*3} - 18a^*b^* + 27c^* = 0$

Let $x = y - a^*/6$, then equation (1) becomes

$$\begin{aligned} y^6 + \left(-\frac{5}{12}a^{*2} + b^*\right)y^4 + \left(\frac{25}{432}a^{*4} - \frac{1}{6}a^{*2}b^* + d^*\right)y^2 \\ + \left(-\frac{1}{81}a^{*5} + \frac{1}{27}a^{*3}b^* - \frac{1}{3}a^*d^* + e^*\right)y \\ + \frac{35}{46656}a^{*6} - \frac{1}{432}a^{*4}b^* + \frac{1}{36}a^{*2}d^* - \frac{1}{6}a^*e^* + f^* = 0. \end{aligned} \tag{2}$$

2.3. In Case of $5a^{*3} - 18a^*b^* + 27c^* \neq 0$

Now apply a Tschirnhausian transform of the following type:

$$-y = p + qx + x^2. \tag{3}$$

Eliminating x from equation (3) and equation (1) with $p = -(a^{*2} - 2b^* - a^*q)/6$, using Sylvester's resultant, gives

$$y^6 + b_0y^4 + c_0y^3 + d_0y^2 + e_0y + f_0 = 0, \tag{4}$$

$$b_0 \equiv 15p^2 + 5py_5 + y_4, \tag{5}$$

$$\begin{aligned} c_0 \equiv 20p^3 + 10p^2 y_5 + 4py_4 + y_3 = \left(-\frac{5}{27}a^{*3} + \frac{2}{3}a^*b^* - c^*\right) q^3 \\ + \left(\frac{5}{9}a^{*4} - \frac{22}{9}a^{*2}b^* + \frac{4}{3}b^{*2} + 3a^*c^* - 4d^*\right) q^2 \\ + \left(-\frac{5}{9}a^{*5} + \frac{26}{9}a^{*3}b^* - \frac{26}{9}a^*b^{*2} - \frac{10}{3}a^{*2}c^* + 3b^*c^* + \frac{13}{3}a^*d^* - 5e^*\right) q \\ + \frac{5}{27}a^{*6} - \frac{10}{9}a^{*4}b^* + \frac{14}{9}a^{*2}b^{*2} - \frac{4}{27}b^{*3} + \frac{4}{3}a^{*3}c^* - \frac{8}{3}a^*b^*c^* \\ + c^{*2} - \frac{4}{3}a^{*2}d^* + \frac{2}{3}b^*d^* + 2a^*e^* - 2f^*, \end{aligned} \tag{6}$$

$$d_0 \equiv 15p^4 + 10p^3 y_5 + 6p^2 y_4 + 3py_3 + y_2, \tag{7}$$

$$e_0 \equiv 6p^5 + 5p^4 y_5 + 4p^3 y_4 + 3p^2 y_3 + 2py_2 + y_1, \tag{8}$$

$$f_0 \equiv p^6 + p^5 y_5 + p^4 y_4 + p^3 y_3 + p^2 y_2 + py_1 + y_0, \tag{9}$$

$$y_5 \equiv -a^*q + a^{*2} - 2b^*, \tag{10}$$

$$y_4 \equiv b^*q^2 + (-a^*b^* + 3c^*)\, q - 2a^*c^* + b^{*2} + 2d^*, \tag{11}$$

$$\begin{aligned} y_3 \equiv -c^*q^3 + (a^*c^* - 4d^*)\, q^2 + (3a^*d^* - b^*c^* - 5e^*)\, q \\ +2a^*e^* - 2b^*d^* + c^{*2} - 2f^*, \end{aligned} \tag{12}$$

$$\begin{aligned} y_2 \equiv d^*q^4 + (-a^*d^* + 5e^*)\, q^3 + (-4a^*e^* + b^*d^* + 9f^*)\, q^2 \\ + (-5a^*f^* + 3b^*e^* - c^*d^*)\, q + 2b^*f^* - 2c^*e^* + d^{*2}, \end{aligned} \tag{13}$$

$$\begin{aligned} y_1 \equiv -e^*q^5 + (a^*e^* - 6f^*)\, q^4 + (5a^*f^* - b^*e^*)\, q^3 \\ + (-4b^*f^* + c^*e^*)\, q^2 + (3c^*f^* - d^*e^*)\, q - 2d^*f^* + e^{*2}, \end{aligned} \tag{14}$$

$$y_0 \equiv f^*q^6 - a^*f^*q^5 + b^*f^*q^4 - c^*f^*q^3 + d^*f^*q^2 - e^*f^*q + f^{*2}. \tag{15}$$

Thus if q is to be one of the roots of the cubic equation $c_0 = 0$, then equation (1) becomes

$$y^6 + b_0 y^4 + d_0 y^2 + e_0 y + f_0 = 0. \tag{16}$$

Therefore, now consider a sextic equation of the form:

$$y^6 + B^* y^4 + D^* y^2 + E^* y + F^* = 0. \tag{17}$$

2.4. In Case of $B^* \neq 0$

Apply a Tschirnhausian transform of the type:

$$-z = p + qy + ry^2 + sy^3 + y^4. \tag{18}$$

Eliminating y from equation (18) and equation (17), using Sylvester's resultant, gives

$$\sum_{n=0}^{6} (p+z)^n G_n(q,r,s) = 0 \quad \text{with} \quad G_6 = 1, \tag{19}$$

$$G_5 \equiv -2B^* r + 2B^{*2} - 4D^*, \tag{20}$$

$$\begin{aligned} G_4 \equiv\, & B^* q^2 + \left(-2B^{*2} + 4D^*\right) qs + \left(B^{*2} + 2D^*\right) r^2 + 5E^* rs \\ & + \left(B^{*3} - 3B^* D^* + 3F^*\right) s^2 + 5E^* q + \left(-2B^{*3} + 2B^* D^* + 6F^*\right) r \\ & - 7B^* E^* s + B^{*4} - 4B^{*2} D^* + 6D^{*2} - 4B^* F^*, \end{aligned} \tag{21}$$

$$\begin{aligned} G_3 \equiv\, & -4D^* q^2 r - 5E^* q^2 s - 5E^* q r^2 + (4B^* D^* - 12F^*) qrs \\ & + 7B^* E^* q s^2 + (-2B^* D^* - 2F^*) r^3 - 3B^* E^* r^2 s \\ & + \left(-2B^{*2} D^* + 2B^* F^* + 4D^{*2}\right) r s^2 + \left(-3B^{*2} E^* + 3D^* E^*\right) s^3 \\ & + (2B^* D^* - 6F^*) q^2 + 4B^* E^* qr + \left(-8D^{*2} + 16B^* F^*\right) qs \\ & + \left(4B^{*2} D^* - 4D^{*2} - 4B^* F^*\right) r^2 + \left(6B^{*2} E^* - 2D^* E^*\right) rs \\ & + \left(5E^{*2} + 2B^* D^{*2} - 4B^{*2} F^* - 2D^* F^*\right) s^2 + \left(B^{*2} E^* - 11D^* E^*\right) q \\ & + \left(-2B^{*3} D^* + 2B^* D^{*2} + 10B^{*2} F^* - 14D^* F^* + 5E^{*2}\right) r \\ & + \left(-3B^{*3} E^* + 6B^* D^* E^* + 11E^* F^*\right) s \\ & - 4B^{*3} F^* + 2B^{*2} D^{*2} + 8B^* D^* F^* - 4B^* E^{*2} - 4D^{*3} + 2F^{*2}. \end{aligned} \tag{22}$$

G_2, G_1, G_0 are polynomials of degree 4, 5, and 6 with respect to the combination of q, r, and s respectively, and are not shown here. Equation (19) can be rearranged as a polynomial of z:

$$z^6 + \sum_{n=0}^{5} H_n z^n = 0, \tag{23}$$

where

$$
\begin{aligned}
H_5 &= 6p + G_5, & (24)\\
H_4 &= 15p^2 + 5pG_5 + G_4, & (25)\\
H_3 &= 20p^3 + 10p^2G_5 + 4pG_4 + G_3, & (26)\\
H_2 &= 15p^4 + 10p^3G_5 + 6p^2G_4 + 3pG_3 + G_2, & (27)\\
H_1 &= 6p^5 + 5p^4G_5 + 4p^3G_4 + 3p^2G_3 + 2pG_2 + G_1, & (28)\\
H_0 &= p^6 + p^5G_5 + p^4G_4 + p^3G_3 + p^2G_2 + pG_1 + G_0. & (29)
\end{aligned}
$$

Since H_5 is linear in p, q, r, s, and H_4 is of second degree, and H_3 is of third degree, one way of determing p, q, r, s to satisfy $H_3 = H_4 = H_5 = 0$ is as follows: at first p, q, r, and s are selected with at least one free parameter so as to satisfy $H_5 = H_4 = 0$, then the parameter is determined from the restriction $H_3 = 0$. That is, let q_0 be one of the roots of the quadratic equation:

$$B^*{q_0}^2 + 5E^*q_0 - \frac{2}{3}B^{*4} + \frac{8}{3}B^{*2}D^* - 4B^*F^* - \frac{2}{3}D^{*2} = 0. \qquad (30)$$

Using q_0, the quantities q_1, r_1, s_1 are defined as

$$q_1 \equiv 2B^*q_0 + 5E^*, \qquad (31)$$

$$r_1 \equiv \frac{4}{3}B^{*3} - \frac{14}{3}B^*D^* + 6F^*, \qquad (32)$$

$$s_1 \equiv \left(-2B^{*2} + 4D^*\right)q_0 - 7B^*E^*. \qquad (33)$$

In case of $q_1 \neq 0$, i.e.

$$25E^{*2} + 4B^*\left(\frac{2}{3}B^{*4} - \frac{8}{3}B^{*2}D^* + 4B^*F^* + \frac{2}{3}D^{*2}\right) \neq 0,$$

$$q \equiv q_0 - (r_1r + s_1s)/\,q_1, \qquad (34)$$

$$r \equiv \lambda\varphi_0, \qquad (35)$$

$$s \equiv \mu\varphi_0, \qquad (36)$$

where λ, μ are eigenvalues corresponding to the homogeneous part (second degree) of $H_4 = 0$. It is necessary only to get the ratio λ/μ or μ/λ through the homogeneous quadratic equation. That is,

$$\left[-\frac{2}{3}B^{*2} + 2D^* + \left(\frac{16}{9}B^{*7} - \frac{112}{9}B^{*5}D^* + \frac{196}{9}B^{*3}D^{*2} + 16B^{*4}F^*\right.\right.$$

$$
\begin{aligned}
&- 56B^{*2}D^*F^* + 36B^*F^{*2}\Big) \Big/ {q_1}^2\Big] \lambda^2 \\
&+ \Big[5E^* + \Big(-\frac{16}{3}B^{*6} + \frac{88}{3}B^{*4}D^* - \frac{112}{3}B^{*2}D^{*2} - 24B^{*3}F^* + 48B^*D^*F^*\Big) \\
&\times q_0/ {q_1}^2 + \Big(-\frac{56}{3}B^{*5}E^* + \frac{196}{3}B^{*3}D^*E^* - 84B^{*2}E^*F^*\Big) \Big/ {q_1}^2 \\
&+ \Big(\frac{8}{3}B^{*5} - \frac{44}{3}B^{*3}D^* + \frac{56}{3}B^*D^{*2} + 12B^{*2}F^* - 24D^*F^*\Big) \Big/ q_1\Big] \lambda\mu \\
&+ \Big[B^{*3} - 3B^*D^* + 3F^* + \Big(4B^{*5} - 16B^{*3}D^* + 16B^*D^{*2}\Big) {q_0}^2/{q_1}^2 \\
&+ \Big(28B^{*4}E^* - 56B^{*2}D^*E^*\Big) q_0/{q_1}^2 + 49B^{*3}E^{*2}/{q_1}^2 \\
&+ \Big(-4B^{*4} + 16B^{*2}D^* - 16D^{*2}\Big) q_0/q_1 \\
&+ \Big(-14B^{*3}E^* + 28B^*D^*E^*\Big) /q_1\Big] \mu^2 = 0\,. \qquad (37)
\end{aligned}
$$

In case of $q_1 = 0$, i.e.

$$25E^{*2} + 4B^*\left(\frac{2}{3}B^{*4} - \frac{8}{3}B^{*2}D^* + 4B^*F^* + \frac{2}{3}D^{*2}\right) = 0,$$

$$q \equiv q_0 + \mu\varphi_0, \qquad (38)$$

$$r \equiv -s_1\lambda\varphi_0, \qquad (39)$$

$$s \equiv r_1\lambda\varphi_0, \qquad (40)$$

$$
\begin{aligned}
B^*\mu^2 + \Big(-2B^{*2} + 4D^*\Big) r_1\mu\lambda + \Big\{ \Big(B^{*3} - 3B^*D^* + 3F^*\Big) {r_1}^2 \\
+ \Big(-\frac{2}{3}B^{*2} + 2D^*\Big) {s_1}^2 - 5E^*r_1s_1\Big\} \lambda^2 = 0. \qquad (41)
\end{aligned}
$$

φ_0 can be determined by $H_3 = 0$ (at most a cubic equation), since all the coefficients appearing in H_3, H_4, and H_5 are rational and each component as a polynomial is irreducible. Finally equation (23) is expressed in the form

$$z^6 + Dz^2 + Ez + F = 0. \qquad (42)$$

2.5. In Case of $B^* = 0$ at equation (17)

Equation (17) itself is already of the form (42).

2.6. In Case of $E = 0$ at equation (42)

Equation (42) itself is a compound cubic equation.

2.7. In Case of $F = 0$ at equation (42)

Equation(42) itself is substantially a quintic equation (or $z = 0$), which is solvable at least using a modular function.

2.8. In Case of $EF \neq 0$ at equation (42)

Let $z \equiv (F/E)w$, then equation (42) becomes

$$w^6 + D\left(\frac{E}{F}\right)^4 w^2 + F\left(\frac{E}{F}\right)^6 (w+1) = 0, \tag{43}$$

which is the final two-parameter equation of the form: $x^6 + ax^2 + bx + b = 0$.

2.9. Solvable Cases of equation (42)

Apply a Tschirnhausian transform

$$-v = p + qz + rz^2 + sz^3 + tz^4 + z^5. \tag{44}$$

Eliminating z from equation (44) and equation (42), using Sylvester's resultant, leads to

$$(v+p)^6 + \sum_{n=0}^{5} (v+p)^n L_n = 0, \tag{45}$$

where

$$L_5 = -4Dt - 5E, \tag{46}$$

$$\begin{aligned} L_4 = {} & 4Dqs + 5Eqt + 2Dr^2 + 5Ers + 6Frt + 3Fs^2 + 6D^2t^2 \\ & + 6Fq - 4D^2s + 11DEt - 5DF + 10E^2, \end{aligned} \tag{47}$$

$$\begin{aligned} L_3 = {} & -4Dq^2r - 5Eq^2s - 6Fq^2t - 5Eqr^2 - 12Fqrs - 8D^2qst \\ & - 11DEqt^2 - 2Fr^3 - 4D^2r^2t + 4D^2rs^2 - 2DErst \\ & + \left(-14DF + 5E^2\right) rt^2 + 3DEs^3 + \left(-2DF + 5E^2\right) s^2t \end{aligned}$$

$$+ 11EFst^2 + \left(-4D^3 + 2F^2\right) t^3 + 8D^2qr - 2DEqs$$
$$+ \left(-4DF - 15E^2\right) qt - DEr^2 + \left(20DF - 15E^2\right) rs - 8EFrt$$
$$- 4EFs^2 + \left(8D^3 + 12F^2\right) st - 7D^2Et^2 - 19EFq + \left(-4D^3 + 6F^2\right) r$$
$$+ 7D^2Es + \left(6D^2F - 9DE^2\right) t + 15DEF - 10E^3. \quad (48)$$

L_2, L_1, and L_0 are polynomials of degree 4, 5, 6 respectively and are not shown here. Rearranging equation (45) as a polynomial of v leads to

$$v^6 + \sum_{n=0}^{5} v^n H_n^* = 0. \quad (49)$$

The forms for H_n^*'s are similar to equations (24)-(29) (replacing H by H^*, and G by L). Let us try to establish $H_5^* = H_3^* = H_1^* = 0$. In this case the resulting equation is a compound cubic equation. From $H_5^* = 0$, we get

$$p = \frac{4Dt + 5E}{6}. \quad (50)$$

Let $q_0 \equiv \frac{5}{3}D - \frac{5}{27}\frac{E^2}{F}$, and q is defined as

$$q \equiv q_0 - \frac{r_1 r + s_1 s + t_1 t}{EF}, \quad (51)$$

where

$$r_1 \equiv -\frac{16}{9}D^3 + 6F^2 + \frac{80}{81}\frac{D^2E^2}{F} - \frac{100}{729}\frac{DE^4}{F^2}, \quad (52)$$

$$s_1 \equiv -\frac{4}{3}D^2E + \frac{80}{81}\frac{DE^3}{F} - \frac{125}{729}\frac{E^5}{F^2}, \quad (53)$$

$$t_1 \equiv -4D^2F + \frac{82}{27}DE^2 - \frac{125}{243}\frac{E^4}{F}. \quad (54)$$

Then we have a one-parameter solution of the form:

$$r = \lambda\varphi_1, \quad (55)$$
$$s = \mu\varphi_1, \quad (56)$$
$$t = \nu\varphi_1, \quad (57)$$

where λ, μ, and ν are eigen values to be determined (only the ratios are required) through the homogeneous parts (second degree and the third degree) of $H_3^* = 0$ under the assumptions (50) and (51), and φ_1 is a parameter to be determined by $H_1^* = 0$. The ratios of eigen values are given, using Sylvester's resultant, as the roots of at most a sextic equation, (since H_3^* is a polynominal of degree 3), therefore solvable cases are obtained by assuming the coefficient of the highest (sixth) degree is equal to zero, which is expressed by

$$\sum_{k,l,m} C_{klm} D^k E^l F^m = 0, \tag{58}$$

where C_{klm} stands for a coefficient and eventually $4k+5l+6m=57, 0 \leq k \leq 16, 0 \leq l \leq 27$, k,l,m: integers. A part (but not all) of the coefficients C_{ijk}'s are

$$C_{13,1,0} = -\frac{573440}{81}, \quad C_{16,1,-2} = -\frac{1310720}{2187}, \quad C_{10,1,2} = -\frac{40960}{3},$$

$$C_{7,1,4} = 46080, \quad C_{15,3,-3} = \frac{508559360}{177147}, \quad C_{12,3,-1} = \frac{2884550656}{59049}$$

In this case the ratios of eigen values, e.g. $\mu/\lambda, \nu/\lambda$ (in case of $\lambda \neq 0$), are determined by at most a quintic equation, and φ_1 by also a quintic equation $H_1^* = 0$.

2.10. Other Solvable Cases of equation (42)

$D = 0$: trinomial equation, roots are expressible in terms of a hypergeometric function. $E^2 - 4DF = 0$: extraction of the square root is possible, resulting in a product of cubic polynomials.

3. Conclusion

A new way for a two-parameter canonical form of sextic equations is given, through a Tschirnhausian transform by multiple steps, supplemented with some soluvable cases.

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