International Journal of Pure and Applied Mathematics

Volume 18 No. 2 2005, 231-233

ON THE PSEUDOCANONICAL MODEL OF A NON-GORENSTEIN INTEGRAL PROJECTIVE CURVE WITH ORDINARY SINGULARITIES

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we compute the numerical invariants of the pseudocanonical model of a non-Gorenstein integral projective curve with ordinary singularities.

AMS Subject Classification: 14H20
Key Words: curves with ordinary singularities, canonical model of a singular curve, non-Gorenstein curve

*

Let X be an integral projective curve of with arithmetic genus $g := p_a(X) \geq 3$ and $f : Y \to X$ its normalization. We assume that X is not Gorenstein, i.e. ω_X s not locally free. We have $h^0(X, \omega_X) = g$. Set $\omega := f^*(\omega_X)/\mathrm{Tors}(f^*(\omega_X)) \in \mathrm{Pic}(Y)$. Since f is an isomorphism outside finitely many points, the natural map $f^*(H^0(X, \omega_X)) \to H^0(Y, \omega)$ is injective and hence its image, Λ, has dimension g. Λ spans ω (see [5]) and hence it induces a morphism $\alpha : Y \to \mathbf{P}^{g-1}$. Set $Z := \mathrm{Im}(\alpha)$ and call $\alpha : Y \to Z$ the surjective morphism induced by α. Thus $Z \subset \mathbf{P}^{g-1}$ is an integral non-degenerate curve. We called it the pseudocanonical model of X. For more on this set-up (see [5] or [3], Section 1.2). From now

Received: November 22, 2004

on we assume that X is not hyperelliptic. This implies that α is birational onto its image ([5], Theorem 17) and that f factors through α, i.e. there is a morphism $\beta : Z \to X$ such that $f = \beta \circ \alpha$ ([5], Theorem 17, or [3], Theorem 1.3). Our aim here is to compute its numerical invariants when X has only ordinary singularities ([1], [2], [4]). For any $P \in X$ let g_P be the arithmetic genus of the singularity (X, P), i.e. the length of the connected component of the sheaf $f_*(\mathcal{O}_Y)/\mathcal{O}_X$ supported by P. For any $P \in X$ let q_P be the length of the connected component of the sheaf $\beta_*(\mathcal{O}_Z)/\mathcal{O}_X$ supported by P. Hence $q_P = g_P - \sum_{Q \in \beta^{-1}(P)} g_Q$.

Remark 1. Fix integers $x \geq n + 1 \geq 2$. Let X be an integral projective curve and $P \in \mathrm{Sing}(X)$ such that the germ (X, P) is formally isomorphic to the germ at $0 \in \mathbb{A}^{n+1}$ of x lines through the origin and not contained in a hyperplane (see [4] for the connection with the notion of ordinary singularities). Let $S \subset \mathbb{P}^n$ be the union of the x points such that (X, P) is formally equivalent to the affine cone of S. For any integer $t \geq 0$ let $\tilde{\rho}_{S,t} : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(S, \mathcal{O}_S(t)) \cong \mathbb{K}^s$ be the restriction map. Set $\rho_{S,t} := \dim(\mathrm{Coker}(\tilde{\rho}_{S,t}))$. We have $g_P = \sum_{t \geq 0} \rho_{S,t}$ ([2], Theorem 5). The conductor of the singularity (X, P) was computed in [4]. Take x distinct points $P_1, \dots, P_x$ of $\mathbb{P}^n$ and set $S := \{P_1, \dots, P_x\}$. We will say that S is "sufficiently general for postulation" if $h^0(\mathbb{P}^n, \mathcal{O}_S(t)) = \max\{0, \binom{n+t}{n} - x\}$ (or, equivalently, $h^1(\mathbb{P}^n, \mathcal{I}_S(t)) = \max\{0, x - \binom{n+t}{n}\}$) for all integers $t > 0$. We will say that S is "general" (for a certain statement) if there is a non-empty Zariski open subset U of the symmetric product of x copies of $\mathbb{P}^n$ for which this statement is true and we take $S \in U$. Notice that $g_P = \sum_{t \geq 0} \max\{0, \binom{n+t}{n} - x\}$ and that this sum is finite. Let $t_{n,x}$ denote the first integer t such that $\binom{n+t}{n} \geq x$. Since X has an ordinary singularity at P if $\alpha(Q_i) = \alpha(Q_1)$ for all i, then β is an isomorphism in a neighborhood of $\alpha(Q_1)$ and $q_P = 0$, while if $\alpha(Q_i) \neq \alpha(Q_j)$ for all $i \neq j$, then α is an isomorphism in a neighborhood of each point Q_i and $q_P = g_P$. Now assume that S is sufficiently general for the postulation. The conductor of the singularity (X, P) (and hence the integer η_P) was computed in [4], part (a) of Theorem 4.4, in which the notation n' instead of n, s instead of x and n instead of $t_{n,x}$ are used. Now assume S general and set $S := \{Q_1, \dots, Q_x\} := f^{-1}(P)$ seen as the union of x distinct points of $\mathbb{P}^n$. By monodromy, the irreducibility of the symmetric product of x copies of P_n and the generality of S we see that if $\alpha(Q_i) = \alpha(Q_j)$ for some $i \neq j$, then $\alpha(Q_i) = \alpha(Q_1)$ for all i and hence β is an isomorphism at $\alpha(Q_1)$. Obviously, this is the case if $x = 2$ (and hence $n = 1$) because an ordinary double point is a Gorenstein singularity, but it is false for every $x \geq 3$ because this is false even for a seminormal singularity with $x \geq 3$ branches.

We summarize the previous work with the following statement.

Theorem 1. *Let X be an integral projective curve with exactly $s > 0$ singular points, say $P_1, \dots, P_s$, each P_i being as described in Remark 1 for some pair of integers (x_i, n_i), $1 \leq i \leq s$. The integers g_{P_i}, $1 \leq i \leq P_i$, are computed in Remark 1. If each set of points $f^{-1}(P_i)$ is sufficiently general for the postulation, then each integer η_{P_i} is computed in [4], part (a) of Theorem 4.4. If $x_i \geq 3$ and the set $f^{-1}(P_i)$ is general, then $q_{P_i} = g_{P_i}$ and α is an isomorphism in a neighborhood of $f^{-1}(P_i)$.*

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] A.V. Geramita, P. Maroscia, The ideal of forms vanishing at a finite set of points in $\mathbb{P}^n$, *J. Algebra*, **90**, No. 2 (1984), 528-555.

[2] A.V. Geramita, F. Orecchia, On the Cohen - Macaulay type of s lines in $\mathbf{A}^{n+1}$, *J. Algebra*, **70**, No. 1 (1981), 116-140.

[3] R.V. Martins, On trigonal non-Gorenstein curves with zero Maroni invariant, *J. Algebra*, **275**, No. 2 (2004), 453-470.

[4] F. Orecchia, Points in generic position and conductors of curves with ordinary singularities, *J. London Math. Soc.*, **24**, No. 2 (1981), 85-96.

[5] M. Rosenlicht, Equivalence relations on algebraic curves, *Ann. of Math.*, **56** (1952), 169-191.