International Journal of Pure and Applied Mathematics
Volume 18 No. 2 2005, 235-239

SCROLLAR INVARIANTS AND SEMINORMAL CURVES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be an integral seminormal projective curve of with arithmetic genus $g := p_a(X) \geq 3$ and $f : Y \to X$ its normalization. Here we study the scrollar invariants of degree k morphism $X \to \mathbb{P}^1$ and extend the related concepts to rational maps (instead of morphisms) and to morphisms with target a smooth curve of genus > 0 instead of $\mathbb{P}^1$.

AMS Subject Classification: 14H20, 14H99
Key Words: scrollar invariants, seminormal curves

1. Scrollar Invariants

Let X be an integral projective curve of with arithmetic genus $g := p_a(X) \geq 3$ and $f : Y \to X$ its normalization. Here we study the scrollar invariants of degree k morphism $X \to \mathbb{P}^1$ (mainly when X has only seminormal singularities) and extend the related concepts to rational maps (instead of morphisms) and to morphisms with target a smooth curve of genus > 0 instead of $\mathbb{P}^1$.

Remark 1. For any integral projective curve D and any degree k morphism $\phi : D \to \mathbb{P}^1$ set $E_\phi := \phi_*(\mathcal{O}_D)/\mathcal{O}_{\mathbb{P}^1})$. Hence E_ϕ is a rank $k-1$ vector bundle on $\mathbb{P}^1$. Let $-a_{1,\phi} \geq \cdots - a_{k-1,\phi}$ be the splitting type of E_ϕ. Using that

Received: November 22, 2004

$R^1\phi_*(\mathcal{O}_D) = 0$ by the finiteness of ϕ and Riemann-Roch on D and $\mathbb{P}^1$ we get $a_{1,\phi} + \cdots + a_{k-1,\phi} = p_a(D) - k + 1$. Similarly, for any smooth curve D, any integral curve X and any degree k morphism $u : X \to D$ set $E_u := u_*(\mathcal{O}_X)/\mathcal{O}_D)$. E_u is a rank $k-1$ vector bundle on D.

Theorem 1. *Fix integers $q \ge 0$, $k \ge 3$, $s > 0$, $k-1 \ge k_1 \ge \cdots \ge k_s \ge 2$ and a smooth and connected projective curve Y with genus q. Assume that one of the following conditions is satisfied:*

(a) $k > q$;

(b) $k_1 = 2$.

Then there exists a projective curve X such that:

(i) *Y is the normalization of X;*

(ii) *X is a seminormal curve with exactly s singular points, say $P_1, \ldots, P_s$, and P_i has exactly k_i branches;*

(iii) *there is a degree k morphism $f : X \to \mathbb{P}^1$ such that the rank $k-1$ vector bundle $E_f := f_*(\mathcal{O}_X)/\mathcal{O}_{\mathbb{P}^1})$ is rigid, i.e. $a_{k-1,f} \ge a_1 1, f - 1$.*

Remark 2. Fix an integer $k \ge 2$, integral projective curves A, B, a degree k morphism $f : A \to B$ and $Q \in B_{reg}$ such that $f^{-1}(Q) \in A_{reg}$ and $\sharp(f^{-1}(Q)) = k$. Let D be curve obtained from A gluing together the k points of $f^{-1}(Q)$ to a seminormal point P of D with k branches. Hence $p_a(D) = p_a(A) + k - 1$. By the universal property of the seminormalization and the smoothness of Q the morphism f induces a degre k morphism $\phi : D \to B$. We have $E_\phi \cong E_f(-Q)$.

Example 1. Fix integers k, q such that $0 \le q \le k-2$, a smooth curve C with genus q and $\mathrm{Pic}^k(C)$ such that $h^1(C, R) = 0$ and R is spanned by its global sections. Since $k > q$ the existence of such a line bundle R is obvious. Fix a linear suspace $V \subseteq H^0(C, R)$ such that $\dim(V) = 2$ and V spans R. let $f : C \to \mathbb{P}^1$ be the degree k morphism induced by the pair (R, V). Let $e_1 \ge \cdots \ge e_{k-1}$ be the scrollar invariants of f. Hence $E_f \cong \oplus_{i=1}^{k-1}\mathcal{O}_{\mathbb{P}^1}(-e_i)$. Since $h^0(C, \mathcal{O}_C) = 1$, $h^0(C, R) = k + 1 - q$, $h^0(C, R^{\oplus 2}) = 2k + 1 - q$ and $p_a(C) = q$ we immediately get $e_i = 2$ if $1 \le i \le q$ and $e_i = 1$ if $q + 1 \le i \le k-1$.

From Example 1 and Remark 2 we immediately get the following result.

Proposition 1. *Fix integers k, q, b such that $0 \le q \le k-2$ and $b \ge 0$ a smooth curve C with genus q. There exist a seminormal curve X and a degree k morphism $f : X \to \mathbb{P}^1$ such that C is the normalization of X, $p_a(C) =$*

$q+b(k-1)$, C has b seminormal points with k branches as only singularities and $E_f \cong \oplus_{i=1}^{k-1}\mathcal{O}_{\mathbb{P}^1}(-e_i)$ with $a_{i,f} = b+2$ if $1 \le i \le q$ and $a_{i,f} = b+1$ if $q+1 \le i \le k-1$.

Proof of Theorem 1. Let A be any integral projective curve and $u : A \to \mathbb{P}^1$ any degree k morphism. The morphism u is uniquely determined by the choice of $R \in \mathrm{Pic}^k(A)$ and a linear subspace V of $H^0(A,R)$ such that $\dim(V) = 2$ and V spans R. The rank $k-1$ vector bundle E_u does not depend from the choice of V (when $h^0(A,R) > 2$), but only from the sequence of integers $\{h^0(A,R^{\otimes t})\}_{t\in\mathbb{N}}$; more precisely, the key integers are the integers $t \ge 2$ such that $h^0(A,R^{\otimes t}) - h^0(A,R^{\otimes(t-1)}) > h^0(A,R^{\otimes(t-1)}) - h^0(A,R^{\otimes(t-2)})$. Fix any $P,Q \in A$ with $P \ne Q$, $Q \in A_{reg}$, $u(P) = u(Q)$ and assume that P is a seminormal point of A with $x \ge 1$ branches. Call A' the curve obtained from A gluing P and Q. Hence there is a birational morphism $v : A \to A'$ such that $v(P) = v(Q)$, $v|A\backslash\{P,Q\}$ is an isomorphism onto $A'\backslash u(P)$ and $u(P)$ is a seminormal singularity of A' with $x+1$ branches. Since $u(P) = u(Q)$ and $v(P)$ is a seminormal singularity of A', the morphism u induces a degree k morphism $u' : A' \to \mathbb{P}^1$. Set $R' := u'^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Assume that $H^0(X,R^{\otimes t})$ separates P and Q, but that $H^0(A,R^{\otimes(t-1)})$ does not separate P and Q. Hence $h^0(A',R'^{\otimes y}) = h^0(A,R^{\otimes y})$ for every $y \ge t-1$ and $h^0(A,R^{\otimes t}) = h^0(A',R'^{\otimes t})+1$. Apply this proof to the pair (A,u) constructed in Proposition 1. □

Example 2. Let C be a smooth and connected projective curve, X an integral projective curve, $f : Y \to X$ the normalization map and $h : Y \to C$ a degree k morphism, i.e. a degree k rational map from Y (or from X) onto C. Set $\mathrm{Sing}(f) : \{P \in X : f \text{ is not regular at } P\}$. Hence $\mathrm{Sing}(f) \subseteq \mathrm{Sing}(X)$. Since $f_*(\mathcal{O}_Y)/\mathcal{O}_X$ is a skyscraper sheaf, there are an integral projective curve D and morphisms $\alpha : Y \to D, \beta : D \to X, \psi : Y \to D, \phi : D \to C$ such that $f = \beta\circ\alpha$, $h = \phi\circ\psi$ and Y has maximal arithmetic genus. The integer $p_a(X) - p_a(D)$ will be called the D-singularity degree of the pair (X,h). Now assume $C = \mathbb{P}^1$. The morphism h is equivalent to a pair (R,V) such that $R \in \mathrm{Pic}^k(Y)$ V is a linear subspace of $H^0(Y,R)$, $\dim(V) = 2$ and V spans R. The coherent sheaf $f_*(R)$ is a rank one torsion free sheaf and $H^0(X,f_*(R)) \cong H^0(Y,R)$ we use this isomorphism to see V as a space of sections of $f_*(\mathcal{O}_Y)$. The evaluation map $\mathcal{O}_Y\otimes V \to R$ induces an evaluation map $\rho : \mathcal{O}_X\otimes V \to f_*(\mathcal{O}_Y)$. Set $\mathcal{F} := \mathrm{Im}(\rho)$. Hence $\mathcal{F}$ is a rank one torsion free sheaf on X, $V \subseteq H^0(X,\mathcal{F})$ and V spans $\mathcal{F}$. Conversely, we start with a rank one torsion free sheaf $\mathcal{G}$ on X and a linear subspace W of $H^0(X,\mathcal{G})$ spanning $\mathcal{G}$ such that $\dim(W) = 2$. Set $\mathrm{Sing}(\mathcal{G}) := \{P \in X : \mathcal{G} \text{ is not locally free at } P\}$. Hence $\mathrm{Sing}(\mathcal{G}) \subseteq \mathrm{Sing}(X)$. Set $L := f^*(\mathcal{G})/\mathrm{Tors}(f^*(\mathcal{G}))$. Since Y is smooth, L is a line bundle. Since tensor product is a right exact functor, W induces a one-dimensional linear subspace W' of $H^0(Y, L)$ spanning L and hence a morphism $h : Y \to \mathbb{P}^1$. It is easy to check that $\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(h)$. From now on in this example we assume $\mathrm{char}(\mathbb{K}) \neq 2, 3$ and that each point of $\mathrm{Sing}(\mathcal{G})$ is either an ordinary node or an ordinary cusp. Let $f' : Y' \to X$ be the partial normalization of X in which we normalize exactly the points of $\mathrm{Sing}(\mathcal{G})$. Hence $p_a(Y') = p_a(X) + \sharp(\mathrm{Sing}(\mathcal{G}))$. Set $M := f'^*(\mathcal{G})/\mathrm{Tors}(f'^*(\mathcal{G}))$. Since $\mathcal{G}$ is locally free outside $\mathrm{Sing}(\mathcal{G})$, the classification of rank one torsion free sheaves at a node or an ordinary cusp gives $M \in \mathrm{Pic}(Y')$, $\mathcal{G} = f'_*(M)$ and $\deg(M) = \deg(\mathcal{G}) - \sharp(\mathrm{Sing}(\mathcal{G}))$. Furthermore, W' spans M and hence it induces a non-constant morphism $h' : Y' \to \mathbb{P}^1$. We will say that (Y', h') is the degree k pencil associated to the pair $(\mathcal{G}, W)$ and its scrollar invariants will be called the scrollar invariants of $(\mathcal{G}, W)$. Notice that neither Y' nor M nor this scrollar invariants depend from the choice of W. The weak scrollar invariants are uniquely determined by the sequence $\{h^0(Y', M^{\otimes t})\}_{t\geq 0}$, i.e. by the sequence $\{h^0(X, f'_*(M^{\otimes t}))\}_{t\geq 0}$. We will call this sequence the associated sequence of $\mathcal{G}$, because it does not depend upon the choice of W.

Example 3. Fix an integer $x \geq 0$, a smooth and connected projective curve C, an integral projective curve D, a degree k morphism $\phi : D \to C$ and $2x$ distinct points $P_i, Q_i \in D_{reg}$, $1 \leq i \leq x$, with different images in C. Let X be the curve obtained from D gluing together each pair $\{P_i, Q_i\}$. The degree k rational map f from X to C induced by ϕ has D-singularity degree x and $\mathrm{Sing}(f)$ is the union of the x nodes obtained by gluing the x pairs $\{P_i, Q_i\}$.

Example 4. Fix an integer $k \geq 3$, a smooth curve C, an integral curve D and finite morphisms $v : C \to \mathbb{P}^1$, $u : Y \to \mathbb{P}^1$, $\deg(u) = k$, such that there is no $P \in \mathbb{P}^1$ which is both the image of a ramification point of u and either a singular point of Y or a ramification point of v. For fixed u, v (as abstract morphisms) we may easily obtain this condition just applying a general element of $\mathrm{Aut}(\mathbb{P}^1)$ to $\mathbb{P}^1$ seen as the target of u. Let X be the fiber product of u and v and $\tilde{u} : X \to C$, $\tilde{v} : X \to D$ the associated pull-back maps. Our assumption on the target $\mathbb{P}^1$ implies that X is reduced. In many cases X is integral; for instance, we may obtain this condition just applying a general element of $\mathrm{Aut}(\mathbb{P}^1)$ to $\mathbb{P}^1$ seen as the target of u. We assume that X is integral. We have $E_{\tilde{u}} = v^*(E_u)$ and hence $E_{\tilde{u}}$ is never stable. Call $e_1 \geq \cdots \geq e_{k-1} > 0$ the scrollar invariants of u. Fix an integer $x \geq 0$ and $2x$ distinct points $P_i, Q_i \in X$, $1 \leq i \leq x$, such that $\tilde{u}(P_i) = \tilde{u}(Q_i)$ for all i. Set $\epsilon := (k-2)e_1 - \sum_{i=2}^{k-1} e_i$. Since E_w is obtained from $E_{\tilde{u}}$ making x negative elementary transformations, E_w is not stable if $x \leq \epsilon + k - 1$ and it is not semistable if either $\epsilon = 0$ and $1 \leq x \leq k-2$

or $\epsilon > 0$ and $0 \leq x < \epsilon$. If $p_a(C) \geq k$, the pairs $\{P_i, Q_i\}$ are sufficiently general and $x \geq \epsilon + k$, then E_w is stable ([1], Corollary 2.4).

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Brill-Noether theory for vector bundles on projective curves, *Math. Proc. Cambridge Phil. Soc.*, **124**, No. 3 (1998), 483-499.