International Journal of Pure and Applied Mathematics

Volume 18 No. 3 2005, 365-370

LOWER BOUNDS FOR THE SUM DIVISOR FUNCTION

Barbara Mędryk

Department of Didactical Mathematics and Number Theory
Faculty of Mathematics, Computer Sciences and Econometrics
University of Zielona Góra
Ul. Professor Szafrana 4a, Zielona Góra, 65-516, POLAND
e-mail: B.Medryk@wmie.uz.zgora.pl

Abstract: Let $\sigma(n)$ be the sum divisor function and let $s(n)$ denote the square-free kernel of positive integer n. We prove that for every k, such that $2 \leq k \leq r = \omega(n)$ we have (*) $\sigma(n) > (\sqrt[r]{n} + \sqrt[r]{n_0})^r \geq (\sqrt[k]{n} + \sqrt[k]{n_0})^k$, where $n_0 = \frac{n}{s(n)}$ and $\omega(n)$ is the number of distinct prime divisor of n. Moreover, we prove that for infinitely many n, we have $\sigma(n) > \frac{6}{\pi^2} e^\gamma n \log\log n$, where $\gamma \approx 0.57721$ is Euler's constant.

AMS Subject Classification: 11A25
Key Words: sum divisor function

1. Introduction

The purpose of this paper is to prove some lower bounds for the sum divisor function $\sigma(n) = \sum\limits_{d|n} d$. Namely, we prove of the following theorem.

Theorem 1. *Let n be a composite positive integer and let $r = \omega(n)$ be the number of all distinct prime divisor of n. Moreover, let $s(n)$ denote the square-free kernel of n. Then for every k such that $2 \leq k \leq r$ we have*

$$\sigma(n) > (\sqrt[r]{n} + \sqrt[r]{n_0})^r \geq (\sqrt[k]{n} + \sqrt[k]{n_0})^k, \tag{*}$$

where $n_0 = \frac{n}{s(n)}$.

Received: December 8, 2004

Immediately from Theorem 1 it follows the following corollary.

Corollary 1. *If* $r = \omega(n) \geq 2$ *then*

$$\sigma(n) > (\sqrt[r]{n} + 1)^r \geq n + 2\sqrt{n} + 1. \tag{**}$$

We note that the inequality (**) is better than classical inequality $\sigma(n) > n + \sqrt{n}$ presented in the Sierpiński's monograph [7], on page 180.

Further, we consider the function $\gamma_n : [1, \infty) \to \mathbb{R}_+$, defined by the rule:

$$\frac{1}{\gamma_n(x)} = x \cdot \left(\left(\frac{\sigma(n)}{n} \right)^{\frac{1}{x}} - 1 \right). \tag{1.1}$$

It is easy to observe that the first derivative of the function (1.1) is negative and moreover the function γ_n increases to $\frac{1}{\log \frac{\sigma(n)}{n}}$ with $t \to \infty$. Hence by the Theorem 1 it follows the following corollary.

Corollary 2. *If* n *is a composite positive integer, then*

$$s(n) > (\gamma_n(r) \cdot r)^r \geq \left(\frac{n}{\sigma(n) - n} \cdot r \right)^r,$$

where $s(n)$ *is the square-free kernel of* n *and* $r = \omega(n)$.

The proof of the Theorem 1 is based on a special version of the Minkowski inequality (see [1], Chapter 2)

$$\prod_{i=1}^{k} (1 + x_i) \geq (1 + \sqrt[k]{x_1 \ldots x_k})^k, \tag{M}$$

where $x_j \geq 0$ for $j = 1, 2, \ldots, k$.

We also note that the inequality (M) has been used by A. Grytczuk and M. Wójtowicz [3] in the proof of an upper bound for the Euler's totient function. We prove also in this paper the following theorem.

Theorem 2. *Let* $\sigma(n)$ *be the sum divisor function. Then for infinitely many positive integers* n, *we have*

$$\sigma(n) > \frac{6}{\pi^2} e^{\gamma} n \log \log n. \tag{***}$$

We note that such type estimation as in (***) is strictly connected with the Riemann Hypothesis. The Riemann zeta function $\zeta(s)$ for $s = \sigma + it$ is defined by Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for $\sigma > 1$ and it has analytic continuation to the complex plane with one singularity as a simple pole and $\underset{s=1}{\text{res}}\zeta(s) = 1$.

In 1859 Riemann [5] conjectured that the nonreal zeros of the Riemann zeta function $\zeta(s)$ all lie on the line $\sigma = \frac{1}{2}$.

The connection of the Riemann Hypothesis with prime numbers has been considered by Gauss.

Let $\pi(x) = \sum_{p \le x} 1$, then it is well-known that the Riemann Hypothesis is equivalent to the assertion that for each $\varepsilon > 0$ there is a positive constant $c = c(\varepsilon)$ such that $|\pi(x) - \text{Li}\,(x)| \le c(\varepsilon)x^{\frac{1}{2}+\varepsilon}$, where $\text{Li}\,(x) = \int\limits_2^x \frac{dt}{\log t}$.

Many others equivalent results with the Riemann Hypothesis are known. We note that more of them has been presented by Conrey in very nice article [2].

We concerne only to one criterion, but very interesting criterion given by Robin [6] in 1984. Robin proved that the Riemann Hypothesis is true if and only if

$$\sigma(n) < e^{\gamma} n \log\log n, \tag{R}$$

for all positive integers $n \ge 5041$, where $\gamma \approx 0.57721$ is the Euler's constant.

2. Proof of Theorem 1

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ then we have

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{1+\alpha_i} - 1}{p_i - 1} = \prod_{i=1}^{r} \left(p_i^{\alpha_i} + p_i^{\alpha_i - 1} + \ldots + p_i + 1\right). \tag{2.1}$$

From (2.1) we have

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{r} \left(1 + \frac{1}{p_i} + \ldots + \frac{1}{p_i^{\alpha_i}}\right). \tag{2.2}$$

By (2.2) it follows that

$$\frac{\sigma(n)}{n} > \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right). \tag{2.3}$$

Putting $x_i = \frac{1}{p_i}$ in the Minkowski inequality (M) from (2.3) we obtain

$$\frac{\sigma(n)}{n} > \prod_{i=1}^{r}\left(1+\frac{1}{p_i}\right) \geq \left(1+\frac{1}{\sqrt[r]{\prod_{i=1}^{r} p_i}}\right)^{r}. \tag{2.4}$$

Since $s(n) = \prod_{i=1}^{r} p_i$ and $n_0 = \frac{n}{s(n)}$ then by (2.4) it follows that

$$\sigma(n) > (\sqrt[r]{n} + \sqrt[r]{n_0})^r, \tag{2.5}$$

and we see that (2.5) proves the first inequality in (*) of Theorem 1.

For the proof of the second part of the inequality in (*) we consider the function: $t \to \left(1+\xi\cdot a^{\frac{1}{t}}\right)^t$ defined on $[1,+\infty)$ for $a \in (0,1)$.

It is easy to observe that this function is increasing for $\xi = 1$. From this fact follows the second inequality in (*) and the proof of Theorem 1 is complete.

3. Proof of Theorem 2.

For the proof of Theorem 2 we consider the following expression:

$$\frac{\sigma(n)\varphi(n)}{n^2}, \tag{3.1}$$

where $\sigma(n) = \sum_{d|n} d$, and $\varphi(n)$ is Euler's totient function. Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$. Then we have

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{1+\alpha_i}-1}{p_i-1} = \frac{n\cdot\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{1+\alpha_i}}\right)}{\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)}, \tag{3.2}$$

$$\varphi(n) = n\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right). \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$\frac{\sigma(n)\varphi(n)}{n^2} = \prod_{i=1}^{r}\left(1-\frac{1}{p_i^{1+\alpha_i}}\right). \tag{3.4}$$

By (3.4) it follows that

$$\frac{\sigma(n)}{n} = \frac{n}{\varphi(n)} \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{1+\alpha_i}}\right). \tag{3.5}$$

Since $\alpha_i \geq 1$ for $i = 1, 2, \ldots, r$ then we have

$$\prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{1+\alpha_i}}\right) \geq \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^2}\right). \tag{3.6}$$

Let P denote the set of all primes, then it is easy to see that

$$\prod_{i=1}^{r} \left(1 - \frac{1}{p_i^2}\right) > \prod_{p \in P} \left(1 - \frac{1}{p^2}\right). \tag{3.7}$$

On the other hand it is well-known that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \prod_{p \in P} \left(1 - \frac{1}{p^2}\right)^{-1}. \tag{3.8}$$

By (3.6)-(3.8) and (3.5) it follows that

$$\frac{\sigma(n)}{n} > \frac{6}{\pi^2} \frac{n}{\varphi(n)}. \tag{3.9}$$

Applying to (3.9) the result given by Nicolas [4] that for infinitely many natural n we have

$$\frac{n}{\varphi(n)} > e^{\gamma} \log\log n\,,$$

we obtain

$$\sigma(n) > \frac{6}{\pi^2} e^{\gamma} n \log\log n,$$

and the proof of Theorem 2 is complete. □

References

[1] Yu.D. Burago, V.A. Zalgaller, *Geometric Inequalities*, Springer-Verlag, Berlin-New York (1988).

[2] J. Brian Conrey, The Riemann Hypothesis, *Notices, Amer. Math. Soc.*, **50**, No. 3 (2003), 341-353.

[3] A. Grytczuk, M. Wójtowicz, An application of the Minkowski inequality, *Int. J. Pure Appl. Math.*, **11**, No. 3 (2004), 311-314.

[4] J.L. Nicolas, Petites valeurs de la fonction d'Euler's, *J. Number Theory*, **17** (1983), 375-388.

[5] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsber. Akad. Berlin* (1859), 671-680.

[6] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothése de Riemann, *J. Math. Pures Appl.*, **63** (1984), 187-213.

[7] W. Sierpiński, *Elemantary Theory of Numbers*, PWN Warszawa (1987).