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ON THE GAUSSIAN MAPS OF BLOWN-UP PROJECTIVE MANIFOLDS

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Abstract: Let S be a smooth projective manifold, $L, M \in \mathrm{Pic}(S)$ and $\gamma_{L,M} : H^0(S,L) \otimes H^0(S,M) \to H^0(S,\Omega_S \otimes L \otimes M)$ the Gaussian (or Wahl) map defined by $\gamma_{L,M}(f \otimes h) = fd(h) - hd(f)$. Here we study the surjectivity of $\gamma_{L,M}$ for suitable L, M when S is the blowing up at finitely many points of a nice manifold (e.g. a projective space). We obtain the surjectivity of suitable Gaussian maps for several subvarieties of S and in particular the surjectivity of $\gamma_{\omega_X,\omega_X}$ for several smooth curves X.

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1. Introduction

Let S be a smooth projective variety and Ω_S its cotangent bundle. For all line bundles L, M on S and all vector spaces $V \subseteq H^0(S,L)$ and $W \subseteq H^0(S,M)$ J. Wahl defined a linear map $\gamma_{V;W} : V \otimes W \to H^0(S,\Omega_S \otimes L \otimes M)$ seeing $f \in V$ (resp. $h \in W$) as a rational function on S and setting $\gamma_{V;W}(f \otimes h) := fd(h) - hd(f)$. If $V = H^0(S,L)$ and $W = H^0(S,M)$ we will write $\gamma_{L,M}$ instead of $\gamma_{V;W}$. If $L = M$ and $V = W$, then $\gamma_{V;V}$ is antisymmetric and hence it

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induces a linear map $\gamma_V : \bigwedge^2(V) \to H^0(S, \Omega_S \otimes L^{\otimes 2})$. If $V = H^0(S, L)$ we will write γ_L instead of γ_V. The linear maps $\gamma_{V;W}$, $\gamma_{L,M}$, γ_V and γ_L are called Gaussian maps or Wahl maps. For several results on the Wahl map γ_{ω_C} of the canonical line bundle of a smooth curve C, see [15]. In [12], Section 6, [16], [4], [5], [7] and [9] there are several computations of Gaussian maps on surfaces and higher dimensional varieties and several applications of these results to the computation of the rank of the Wahl map of the canonical line bundle of certain smooth curves.

In Section 3 we will prove the following result on Gaussian maps on blown-up surfaces and two related results (see Theorem 5 and Proposition 1).

Theorem 1. *Let X be a smooth and connected projective surface, A, B very ample line bundles on X, $P \in X$ and $T \subset X$ a smooth connected curve with $P \in T$. Let $f : S \to X$ be the blowing-up of X at P and $D \subset S$ the strict transform of T. Set $g := p_a(T)$ and $E := f^{-1}(P)$. Fix integers $a \geq 0$, $b \geq 0$ and set $M := f^*(A)(-aE)$ and $N := f^*(B)(-bE))$. Assume that the following conditions are satisfied:*

(a) the Gaussian map $\gamma_{A,B}$ on X is surjective;

(b) $H^1(X, A(-iT)) = H^1(X, B(-jT)) = 0$ for all i, j, such that $0 \leq i \leq a+1$ and $0 \leq j \leq b+1$;

(c) $A \cdot T \geq a|T^2 - 1| + 2g + 2$, $B \cdot T \geq b|T^2 - 1| + 2g + 2$ and $(A + B) \cdot T \geq (a + b)|T^2 - 1| + 6g + 3$.

Then the Gaussian map $\gamma_{M,N}$ on S is surjective.

In Section 4 we will consider a general blowing-up of a projective space. For all positive integers n, t define the integer $x_{n,t}$ in the following way. Set $x_{n,1} := 0$ for every $n \geq 1$ and $x_{1,t} := [t/2]$ for every $t \geq 1$. Define inductively the integer $x_{n,t}$, $n \geq 2$, $t \geq 2$, by the formula $x_{n.t} = x_{n-1,t} + x_{n,t-1}$. Hence $\binom{n+t-2}{n}/2 \leq x_{n,t} \leq \binom{n+t-1}{n}/2$ for all n, t. In Section 4 we will prove the following result.

Theorem 2. *Fix integers n, r, t, x, y such that $n \geq 1$, $r \geq t \geq 1$ and $0 \leq y \leq x \leq x_{n,t}$. Let $A \subset \mathbf{P}^n$ be a general set with $\mathrm{card}(A) = x$ and $f : S \to \mathbf{P}^n$ the blowing-up of A. Take any subset A' of A with $\mathrm{card}(A') = y$. Let $\Delta \subset S$ be the union of all exceptional divisors of f and $\Delta' \subseteq \Delta$ the union of the exceptional divisors over the points of A'. For each integer z set $L(z, A', A) := f^*(\mathcal{O}_{\mathbf{P}^n}(z))(-\Delta - \Delta') \in \mathrm{Pic}(S)$. Then the Gaussian map $\gamma_{L(t,A',A),L(r,A',A)}$ is surjective.*

To prove Theorem 2 we will use some results on the surjectivity of the multiplication map $H^0(\mathbf{P}^n, \mathcal{I}_A(t)) \otimes H^0(\mathbf{P}^n, \mathcal{I}_A(r)) \to H^0(\mathbf{P}^n, (\mathcal{I}_A)^2(t+r))$ (see Theorem 6). In Section 5 we will show the surjectivity of the Wahl map for the canonical line bundle of smooth curves arising as normalization of an integral nodal curve $C_t \subset \mathbf{P}^3$ with C_t complete intersection of a smooth quintic surface and another surface of degree $t \geq 15$. In this way we will obtain a cheap proof not using degeneration theory and reducible curves of the following result.

Theorem 3. *Let X be a general smooth curve of genus $g \geq 601$. Then the Wahl map γ_{ω_X} is surjective.*

The optimal range ($g \geq 12$ and $g = 10$) was proved in [6] using a stable genus g curve with smooth rational curves as irreducible components. For infinitely many integers g the surjectivity of the Wahl map of the canonical line bundle of a general genus g smooth curve was proved in [12] in a very nice way: just showing the surjectivity of the Wahl map of the canonical line bundle for almost all complete intersection curves. To prove Theorem 3 we will prove the following result.

Theorem 4. *Fix integers t, w with $t \geq 15$ and $0 \leq w \leq 5t-1$. Let $S \subset \mathbf{P}^3$ be a smooth quintic surface and a general $A \subset S$ such that $\mathrm{card}(A) = w$. Let $f : W \to S$ be the blowing-up of A and $E := f^{-1}(A)$ the excetional divisor of A. Set $L := f^*(\mathcal{O}_{\mathbf{P}^3}(t+1)) \otimes \mathcal{O}_W(-E) \in \mathrm{Pic}(W)$. Then γ_L is surjective.*

2. The Multiplication Map

In this section we will connect the surjectivity of a Gaussian map for a line bundle on a projective manifold S to the surjectivity of the multiplication map for certain line bundles on an effective smooth Cartier divisor D of S, a Gaussian map on D and a Gaussian map for "smaller" line bundles on S. This will be essentialy the computation of a residue.

Remark 1. Let S be a smooth projective variety and $D \subset S$ a smooth effective Cartier divisor. The line bundle $\mathcal{O}_D(-D)$ is the conormal bundle to D in S and we have an exact sequence on D:

$$0 \to \mathcal{O}_D(-D) \to \Omega_S|D \to \Omega_D \to 0\,. \tag{1}$$

Fix $L, M \in \mathrm{Pic}(S)$ and linear subspaces $V \subseteq H^0(S, L)$ and $W \subseteq H^0(S, M)$. Let $\rho_{L,D} : H^0(S, L) \to H^0(D, L|D)$ and $\rho_{M,D} : H^0(S, M) \to H^0(D, M|D)$ denote the restriction map. Set $V' := \rho(L, D)(V) \subseteq H^0(D, L|D)$ and $W' :=$

$\rho_{M,D}(W) \subseteq H^0(S, M|D)$. Let $\zeta : V \otimes W \to V' \otimes W'$ be the restriction map. The functoriality of the cotangent sheaf induces a linear map $\tau : H^0(S, \Omega_S \otimes L \otimes M) \to H^0(D, \Omega_D \otimes (L|D) \otimes (M|D))$ such that $\tau \circ \gamma_{V;W} = \gamma_{V';W'} \circ \zeta$. The map τ was used in several cases to compute the rank of the Gaussian map for D a smooth curve contained in a smooth surface S in the case $L|D \cong \omega_D$ (see e.g. [12], Theorem 6.2, [4] and [5]). Fix $P \in D$ and let z be a local equation of D in S around P. Hence locally around P we have $\Omega_D \cong \Omega_S/(z, dz)|D$. For any $f \in H^0(S, L)$, $h \in H^0(S, M(-D))$ we have $fd(zh) - zhd(f) = zfd(h) + fhd(z) - zhd(f)$. The restriction of $zfd(h) - zhd(f)$ to D vanishes identically. The map $\rho : \Omega_S|D \to \Omega_D$ appearing in (1) sends the term $fhd(z)$ into zero and hence the term $fhd(z)$ corresponds to an element of $H^0(D, (L|D) \otimes (M(-D)|D))$.

Lemma 1. *Let S be a smooth projective variety and $D \subset S$ a smooth effective Cartier divisor. Fix $L, M \in \mathrm{Pic}(S)$ and linear subspaces $V \subseteq H^0(S, L)$ and $W \subseteq H^0(S, M)$. Call V' (resp. W') the image of V (resp. W) by the restriction map $H^0(S, L) \to H^0(D, L|D)$ (resp. $H^0(S, M) \to H^0(D, M|D)$). Set $W_1 := W \cap H^0(S, M \otimes \mathcal{I}_D)$ and see W_1 as a linear subspace of $H^0(S, M(-D))$. Call W'' the image of W_1 by the restriction map $H^0(S, M(-D)) \to H^0(D, M(-D)|D)$. Assume the surjectivity of the Gaussian maps $\gamma_{V;W_1}$ (on S) and $\gamma_{V';W'}$ (on D) and of the multiplication map $V' \otimes W'' \to H^0(D, L \otimes M(-D)|D)$. Then $\gamma_{V;W}$ is surjective.*

Proof. Let $\alpha : H^0(S, \Omega_S \otimes L \otimes M) \to H^0(D, (\Omega_S \otimes L \otimes M)|D)$ be the restriction map. The surjectivity of $\gamma_{V;W_1}$ implies $\mathrm{Ker}(\alpha) \subseteq \mathrm{Im}(\gamma_{V;W})$. Hence it is sufficient to show the surjectivity of the map $\alpha \circ \gamma_{V;W}$. The surjectivity of $\gamma_{V';W'}$ implies the surjectivity of $\eta \circ \alpha \circ \gamma_{V,W}$, where $\eta : H^0(D, (\Omega_S \otimes L \otimes M)|D) \to H^0(D, \Omega_D \otimes (L \otimes M)|D)$ is the map induced by the conormal exact sequence (1). Notice that the map $f \otimes h \mapsto fh$ considered in Remark 1 is induced from the multiplication map $H^0(D, L|D) \otimes H^0(D, M(-D)|D) \to H^0(D, L \otimes M(-D)|D)$. Hence $\mathrm{Ker}(\eta) \subseteq \mathrm{Im}(\eta \circ \alpha \circ \gamma_{V,W})$ by the surjectivity of the multiplication map $V' \otimes W'' \to H^0(D, L \otimes M(-D)|D)$, concluding the proof. □

Corollary 1. *Let S be a smooth projective variety, $D \subset S$ a smooth effective Cartier divisor and $L, M \in \mathrm{Pic}(S)$. Assume the surjectivity of the restriction maps $H^0(S, L) \to H^0(D, L|D)$, $H^0(S, M) \to H^0(D, M|D$, $H^0(S, M(-D)) \to H^0(D, M(-D))$, of the multiplication map $H^0(D, L|D) \otimes H^0(D, M(-D)|D) \to H^0(D, L \otimes M(-D)|D)$ and of the Gaussian maps $\gamma_{L,M(-D)}$ and $\gamma_{L|D,M|D}$. Then $\gamma_{L,M}$ is surjective.*

3. Blown-up Surfaces

Proof of Theorem 1. First assume $a = b = 0$. Since $f_*(\mathcal{O}_S) = \mathcal{O}_X$, we have $f_*(M) \cong A$, $f_*(N) \cong B$ (projection formula) and hence $H^0(S, M) \cong H^0(X, A)$, $H^0(S, N) \cong H^0(X, B)$. The functoriality of the Wahl map implies the existence of a commutative diagram

$$\begin{array}{ccc} H^0(S, M) \otimes H^0(S, N) & \overset{\gamma_{M,N}}{\longrightarrow} & H^0(S, \Omega_S \otimes M \otimes N) \\ \uparrow \psi_1 & & \uparrow \psi_2 \\ H^0(X, A) \otimes H^0(X, B) & \overset{\gamma_{A,B}}{\longrightarrow} & H^0(X, \Omega_X \otimes A \otimes B) \end{array}$$

in which ψ_1 is the isomorphism induced by f^*. Just looking to $S \backslash E \cong X \backslash \{P\}$ we see the injectivity of ψ_2. Since $\gamma_{A,B}$ is surjective, it is sufficient to show that ψ_2 is surjective. The coherent sheaf $\Omega_S \otimes M \otimes N / f^*(\Omega_X \otimes A \otimes B) \cong (\Omega_S / f^*(\Omega_X)) \otimes f^*(A \otimes B) \cong \Omega_S / f^*(\Omega_X)$ is an $\mathcal{O}_E$-sheaf isomorphic to $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbf{P}^1}(-1)$. Thus $H^0(S, \Omega_S \otimes M \otimes N / f^*(\Omega_X \otimes A \otimes B)) = 0$. Hence the injectivity of ψ_2 implies its surjectivity. Now we assume $a + b > 0$, say $b > 0$. We use induction on the integer $a + b$. To apply Corollary 1 it is sufficient to check the following facts:

(i) The surjectivity of $\gamma_{M,N(-D)}$. Since $N(-D) \cong f^*(B - T)(-(b-1)D)$, this follows from the inductive assumption for the data $(A, B - T, a, b - 1)$ because this quadruple satisfy all the assumption of the theorem; in particular notice that it satisfies condition (b).

(ii) The surjectivity of the Wahl map $\gamma_{M|D,N|D}$.

(iii) The surjectivity of the restriction maps $H^0(S, M) \to H^0(D, M|D)$ and $H^0(S, N) \to H^0(D, N|D)$. Use assumption (b) and the equalities $h^1(S, M(-tD)) = h^1(X, A(-tT))$ and $h^1(S, N(-tD)) = h^1(X, B(-tT))$ which are true for every integer $t \geq 0$ because $R^0 f_*(M(-tD)) \cong A(-tD)$, $R^1 f_*(M(-tD)) \cong A(-T) \otimes R^0 f_*(\mathcal{O}_S) \cong A(-tD)$ and similarly for N, plus the Leray spectral sequence of f.

(iv) The surjectivity of the Gaussian map $\gamma_{M|D,N|D}$ on D and of the multiplication map $\mu : H^0(D, L|D) \otimes H^0(D, M(-D)|D) \to H^0(D, L \otimes M(-D)|D)$. We have $\deg(M|D) = A \cdot T$ and $\deg(N|D) = B \cdot T$. Since P is a smooth point of T, we have $D \cong T$ and $D^2 = T^2 - 1$. Apply [3], Theorem 1, part (i), to the smooth genus g curve D and assumption (c) to obtain the surjectivity of $\gamma_{M|D,N|D}$. By a very particular case of [8], Theorem 1, μ is surjective.

Theorem 5. *Fix an integer $k \geq 1$ and non-negative integers a_i, b_i, $1 \leq i \leq k$. Let X be a smooth and connected projective surface and $P_1, \ldots, P_k$*

distinct points of X. Let $f : S \to X$ be the blowing-up of $P_1, \dots, P_k$ and $E_i := f^{-1}(P_i)$, $1 \le i \le k$, the exceptional divisors. Fix very ample line bundles A, B, H on X and set $M := f^*(A)(-\sum a_i E_i)$, $N := f^*(B)(-\sum b_i E_i)$. Let $g := H \cdot (K_X + H)/2 + 1$ be the sectional genus of H. Assume that the following conditions are satisfied:

(a) *the Gaussian map $\gamma_{A,B}$ on X is surjective;*

(b) $H^1(X, A(-iH)) = H^1(X, B(-jH)) = 0$ *for all i, j, such that $0 \le i \le a+1$ and $0 \le j \le b+1$;*

(c) $A \cdot T \ge (\sum_i a_i)(H^2 - 1) + 2g + 2$, $B \cdot T \ge (\sum_i b_i)(H^2 - 1) + 2g + 2$ *and* $(A + B) \cdot T \ge (\sum_i (a_i + b_i))(H^2 - 1) + 6g + 3$.

Then the Gaussian map $\gamma_{M,N}$ on S is surjective.

Proof. Since H is very ample, for every $i \in \{1, \dots, k\}$ there is a smooth and connected $T_i \in |H|$ such that $P_i \in T_i$ and $P_j \notin T_i$ for all $j \ne i$. Hence we may iterate k times the proof of Theorem 1. □

The same proof gives the following result.

Proposition 1. *Fix integers $a \ge 0$, $b \ge 0$ and $k \ge 1$. Let X be a smooth and connected projective surface, A, B and H very ample line bundles on T, $P_1, \dots, P_k$ distinct points of X such that there is a smooth $T \in |H|$ with $P_i \in T$ for every i. Let $f : S \to X$ be the blowing-up of X at $P_1, \dots, P_k$, $D \subset S$ the strict transform of T and $E_i := f^{-1}(P_i)$. Set $M := f^*(A)(-\sum aE_i)$ and $N := f^*(-\sum bE_i)$. Let $g := H \cdot (K_X + H)/2 + 1$ be the sectional genus of H. Assume that the following conditions are satisfied:*

(a) *the Wahl map $\gamma_{A,B}$ on X is surjective;*

(b) $H^1(X, A(-iT)) = H^1(X, B(-jT)) = 0$ *for all i, j, such that $0 \le i \le ka+1$ and $0 \le j \le kb+1$;*

(c) $A \cdot T \ge a|T^2 - k| + 2g + 2$, $B \cdot T \ge a|T^2 - k| + 2g + 2$ *and* $(A + B) \cdot T \ge (a + b)|T^2 - k| + 6g + 3$. *Then the Wahl map $\gamma_{M,N}$ on S is surjective.*

4. Blowning-up the Projective Space

Remark 2. Fix positive integers r, t, n and set $L := \mathcal{O}_{\mathbf{P}^n}(r)$ and $M := \mathcal{O}_{\mathbf{P}^n}(t)$. By [12], Theorem 6.4, the map γ_L is surjective. More generally, using [12], Remark 6.7.3, it is easy to check that $\gamma_{L,M}$ is surjective even if $r \neq t$ ([13], 9.7.9, or [14], 1.3.2). A more general result on an arbitrary flag variety was conjectured in [16], p. 231. A proof of Wahl's conjecture was announced in [10].

Lemma 2. *Fix positive integers n, t, x such that $n \geq 1$, $t \geq 2$, $0 \leq y \leq x$, $x - y + (n+1)y \leq \binom{n+t-2}{n}$ and*

$$(n+1)y + n^2t^2 \leq \binom{n+t-2}{n}. \tag{2}$$

Let $E \subset \mathbf{P}^n$ be a general subset such that $\operatorname{card}(E) = x$ *and $A \subseteq E$ such that* $\operatorname{card}(A) = y$. *Set $B := E \backslash A$. Then $h^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(t) \otimes \mathcal{I}_B \otimes (\mathcal{I}_A)^2) = 0$.*

Proof. We have $h^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(t)) = 0$. Thus $h^1(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(t) \otimes \mathcal{I}_B \otimes (\mathcal{I}_A)^2) = 0$ if and only if $h^0(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(t) \otimes \mathcal{I}_B \otimes (\mathcal{I}_A)^2) = h^0(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(t)) - n(n+1)y - n(x-y)$. Since $\Omega_{\mathbf{P}^n}(2)$ is spanned by its global sections, $\Omega_{\mathbf{P}^n}(t)$ contains a subsheaf $F \cong \mathcal{O}_{\mathbf{P}^n}(t-2)$. Thus it is sufficient to show that $h^0(\mathbf{P}^n, F \otimes \mathcal{I}_B \otimes (\mathcal{I}_A)^2) = h^0(\mathbf{P}^n, F) - n(n+1)y - n(x-y)$. Hence it is sufficient to prove that $h^0(\mathbf{P}^n, \mathcal{I}_B \otimes (\mathcal{I}_A)^2(t-2)) = h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t-2)) - (n+1)y - (x-y)$. Since $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t-2)) \geq (n+1)y + (x-y)$ and B is general, it is sufficient to prove that $h^1(\mathbf{P}^n, (\mathcal{I}_A)^2(t-2)) = 0$. This is a very particular case of [1]. □

Remark 3. Fix a hyperplane H of $\mathbf{P}^n$ and take integers x, y and t as in Lemma 2. With our very strong restriction given by the inequality (2) the proof of [1] and hence of Lemma 2 works if instead of a general $S \subset \mathbf{P}^n$ we take a set containing $\binom{n+t-4}{n-1}$ points of H and as general as possible with this restriction. It is for this crucial observation that we assumed the inequality (2). For our applications to Theorem 2 this restriction is not expensive.

Theorem 6. *Fix integers n, r, t, x such that $n \geq 1$, $r \geq t \geq 2$ and $0 \leq x \leq \binom{n+t-2}{n}$. For a general $A \subset \mathbf{P}^n$ such that* $\operatorname{card}(A) = x$ *we have $h^1(\mathbf{P}^n, \mathcal{I}_A(t)) = h^1(\mathbf{P}^n, \mathcal{I}_A(r)) = h^1(\mathbf{P}^n, (\mathcal{I}_A)^2(r+t)) = 0$ and the multiplication map $\mu_{A,r,t,n}$: $H^0(\mathbf{P}^n, \mathcal{I}_A(t)) \otimes H^0(\mathbf{P}^n, \mathcal{I}_A(r)) \to H^0(\mathbf{P}^n, (\mathcal{I}_A)^2(r+t))$ is surjective.*

Proof. Since $x \leq h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) \leq h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(r))$ and A is general, we have $h^1(\mathbf{P}^n, \mathcal{I}_A(t)) = h^1(\mathbf{P}^n, \mathcal{I}_A(r)) = 0$. By [1], Theorem 0.1, we have $h^1(\mathbf{P}^n, (\mathcal{I}_A)^2(r+t)) = 0$. Hence it is sufficient to prove the surjectivity of

$\mu_{A,r,t,n}$. The case $n = 1$ is obvious because if $n = 1$ the sheaves $\mathcal{I}_A$ and $(\mathcal{I}_A)^2$ are line bundles respectively of degree $-x$ and $-2x$. Thus we may assume $n \geq 2$ and that the result is true in $\mathbf{P}^{n-1}$. We may also use induction on the integer t. The result is obviously true if $t = 2$. Hence we may assume $t \geq 3$ and that the result is true in $\mathbf{P}^n$ for all triples (t', r', x') such that $t' < t$, $r' \geq t'$ and $0 \leq x' \leq \binom{n+t'-2}{n}$. First we assume $x \geq \binom{n+t-3}{n-1}$. Fix a hyperplane H of $\mathbf{P}^n$, a general $B \subset \mathbf{P}^n$ with $\mathrm{card}(B) = x - \binom{n+t-3}{n-1}$ and a general $E \subset H$ with $\mathrm{card}(E) = \binom{n+t-3}{n-1}$. It is easy to show using Horace Method that $h^1(\mathbf{P}^n, \mathcal{I}_{B\cup E}(t-1)) = h^1(\mathbf{P}^n, \mathcal{I}_{B\cup E}(r-1)) = h^1(\mathbf{P}^n, (\mathcal{I}_{B\cup E})^2(r+t)) = 0$ (Remark 3). Hence by semicontinuity to prove Theorem 6 it is sufficient to prove the surjectivity of the multiplication map $\mu_{B\cup E,r,t,n}$. For any integer u and any subscheme Γ of $\mathbf{P}^n$ let $\alpha_{\Gamma,u} : H^0(\mathbf{P}^n, \mathcal{I}_\Gamma(u)) \to H^0(H, \mathcal{I}_\Gamma \otimes \mathcal{O}_H(u))$ denotes the restriction map. The maps $\alpha_{B\cup E,t}$ and $\alpha_{B\cup E,r}$ are surjective because $h^1(\mathbf{P}^n, \mathcal{I}_{B\cup E}(t-1)) = h^1(\mathbf{P}^n, \mathcal{I}_{B\cup E}(r-1)) = 0$ (use Horace Lemma). The residual scheme of $2B \cup 2E$ with respect to H is the scheme $2B \cup E$. We have $\mathrm{Ker}(\alpha_{B\cup E,r}) \cong H^0(\mathbf{P}^n, \mathcal{I}_B(r-1))$ and $\mathrm{Ker}(\alpha_{B\cup E,t}) \cong H^0(\mathbf{P}^n, \mathcal{I}_B(t-1))$. Thus $\mathrm{Ker}(\alpha_{2B\cup 2E,r+t}) \cong H^0(\mathbf{P}^n, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(r+t-1))$; more precisely, if $z = 0$ is an equation of H, then $\mathrm{Ker}(\alpha_{2B\cup 2E,r+t})$ is given by the homogeneous polynomials of the form zf with $f \in H^0(\mathbf{P}^n, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(r+t-1))$. By [1], Theorem 0.1, we have $h^1(\mathbf{P}^n, (\mathcal{I}_B)^2(r+t-2)) = 0$. Since E is general in H, we obtain $h^1(\mathbf{P}^n, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(r+t-1)) = 0$. Thus $\alpha_{2B\cup 2E,r+t}$ is surjective. By the inductive assumption on n the multiplication map $\mu_{E,r,t,n-1} : H^0(H, \mathcal{I}_E(t)) \otimes H^0(H, \mathcal{I}_E(t)) \to H^0(H, (\mathcal{I}_E)^2(r+t))$ is surjective. Hence to prove Theorem 6 it is sufficient to prove $\mathrm{Ker}(\alpha_{2B\cup 2E,r+t}) \subseteq \mathrm{Im}(\mu_{B\cup E,r,t,n})$. Call V (resp. W) the subspace of $H^0(\mathbf{P}^n, \mathcal{I}_{B\cup E}(r))$ (resp. $H^0(\mathbf{P}^n, \mathcal{I}_{B\cup E}(t))$) formed by all polynomials vanishing on H. Thus $V \cong H^0(\mathbf{P}^n, \mathcal{I}_B(r-1))$ and $W \cong H^0(\mathbf{P}^n, \mathcal{I}_B(t-1))$. We have $\mu_{2B\cup 2E,r,t,n}(H^0(\mathbf{P}^n, \mathcal{I}_{B\cup E}(r)) \otimes W) \subseteq \mathrm{Ker}(\alpha_{2B\cup 2E,r+t})$. By the inductive assumption on t the multiplication map $\mu_{B,r-1,t-1,n}$ is surjective, i.e. the multiplication map $V \otimes W \to H^0(\mathbf{P}^n, (\mathcal{I}_B)^2 \otimes (\mathcal{I}_H)^2(r+t))$ is surjective. Hence to prove the theorem it is sufficient to prove the surjectivity of the multiplication map $\beta_{B,E,r-1,t} : H^0(\mathbf{P}^n, \mathcal{I}_B(t-1)) \otimes H^0(\mathbf{P}^n, \mathcal{I}_{B\cup E}(r)) \to H^0(\mathbf{P}^n, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(r+t-1))$. The restriction map $\psi : H^0(\mathbf{P}^n, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(r+t-1)) \to H^0(H, \mathcal{I}_E(t+r-1))$ is surjective and $\mathrm{Ker}(\psi) \cong H^0(\mathbf{P}^n, (\mathcal{I}_B)^2(t+r-2))$. Since $\alpha_{B\cup E,r}$ is surjective, the surjectivity of $\beta_{B,E,r-1,t}$ follows from the surjectivity of ψ and the surjectivity of $\mu_{B,r-1,t-1,n}$. The case $x < \binom{n+t-3}{n-1}$ is similar: just take a general $E \subset H$ such that $\mathrm{card}(E) = x$ and $B = \emptyset$. □

Remark 4. Fix integers n, r, t, x, y such that $n \geq 2$, $r \geq t \geq 2$, $0 \leq x \leq \binom{n+t-2}{n}$, $0 \leq y \leq \binom{n+t-3}{n-1}$ and $y \leq x$. Fix a hyperplane H of $\mathbf{P}^n$.

The proof of Theorem 6 and semicontinuity shows the existence of $A \subset \mathbf{P}^n$ such that $\mathrm{card}(A) = x$, $\mathrm{card}(A \cap H) = y$, $h^1(\mathbf{P}^n, \mathcal{I}_A(t)) = h^1(\mathbf{P}^n, \mathcal{I}_A(r)) = h^1(\mathbf{P}^n, (\mathcal{I}_A)^2(r+t)) = 0$ and the multiplication map $\mu_{A,r,t,n} : H^0(\mathbf{P}^n, \mathcal{I}_A(t)) \otimes H^0(\mathbf{P}^n, \mathcal{I}_A(r)) \to H^0(\mathbf{P}^n, (\mathcal{I}_A)^2(r+t))$ is surjective.

Proof of Theorem 2. The case $x = y = 0$ is true by [13], 9.7.9, or [14], 1.3.2. Hence we may apply induction on the integer $x + y$. Since $r \ge t \ge 2x \ge x + y$, the case $n = 1$ is true: in this case $S = \mathbf{P}^1$ and $L(z, A', A)$ is a line bundle of degree $z - x - y$. Hence we may assume $n \ge 2$ and that the result is true for the integer $n - 1$. Fix a hyperplane H of $\mathbf{P}^n$. First assume $y \ge x_{n-1,t}$. Take a general $E \subset H$ with $\mathrm{card}(E) = x_{n-1,t}$, a general $B \subset \mathbf{P}^n \backslash H$ with $\mathrm{card}(B) = x - x_{n-1,t}$ and any $B' \subseteq B$ with $\mathrm{card}(B') = y - x_{n-1,t}$. Set $A := E \cup B$ and $A' := E \cup B'$. Abusing notation, call $f : S \to \mathbf{P}^n$ the blowing-up of A. Set $\Delta(A') := f^{-1}(A')$ and $\Delta(A) := f^{-1}(A)$. Call $f_H : D \to H$ the blowing-up of H along E. Hence D is the strict transform of H in S. Set $\Delta_H := f_H^{-1}(E)$. For all finite subsets G, G' of E with $G' \subseteq G$ set $L_H(z, G', G) := f_H^*(\mathcal{O}_H(z))(-f_H^{-1}(G') - f_H^{-1}(G))$. Under our assumption on y we have $L_H(z, A', E) = f_H^*(\mathcal{O}_H(z))(-f_H^{-1}(A') - f_H^{-1}(E))$. Notice that D is the strict transform of H in S and $D \in |L(1, \emptyset, E)|$. By [1], Theorem 0.1, we have $h^1(S, L(t, A', A)) = h^1(S, L(r, A', A)) = 0$. By Lemma 2 we have $H^1(S, \Omega_S \otimes L(t, A', A)) = 0$. Thus by semicontinuity it is sufficient to prove the surjectivity of the map $\gamma_{L(t,A',A),L(r,A',A)}$. We have $L(z, A', A)|D \cong L_H(z, E, E)$ and $L(t, A', A)(-D)|D \cong L_H(t-1, \emptyset, E)$. Since $H^1(S, L(t-1, \emptyset, B)) = H^1(S, L(r-1, \emptyset, B)) = 0$, the restriction maps $H^0(S, L(t, A', A)) \to H^0(D, L(t, A', A)|D)$ and $H^0(S, L(r, A', A)) \to H^0(D, L(r, A', A)|D)$ are surjective. The surjectivity of the map $\gamma_{L,M(-D)}$ in the statement of Corollary 1 is satified by the inductive assumption on the integer $x + y$. By Corollary 1 it is sufficient to prove the surjectivity of the multiplication map $H^0(D, L_H(t-1, \emptyset, E)) \otimes H^0(D, L_H(r, \emptyset, E)) \to H^0(D, L_H(r+t-1, E, E))$. This is given by Theorem 6 applied to the integer $n' := n - 1$. Now assume $y < x_{n-1,t} \le x$. Let $E \subset H$ be a general subset with $\mathrm{card}(A) = x_{n-1,t}$. Fix any $A' \subset E$ with $\mathrm{card}(A') = y$. Take a general subset B of $\mathbf{P}^n$ with $\mathrm{card}(B) = x - x_{n-1,t}$ and set $A := E \cup B$, $\Delta := f_H^{-1}(E)$, $\Delta' := f_H^{-1}(A')$ and $L_H(z, E', E) := f_H^*(\mathcal{O}_H(z))(-\Delta - \Delta')$. Then we repeat the previous proof. A similar proof works if $x < x_{n-1,t}$, just taking $E \subset H$ with $\mathrm{card}(E) = x$ and $B = \emptyset$. □

5. Proofs of Theorem 3 and Theorem 4

In this section we will prove Theorem 3 and Theorem 4.

Remark 5. Let $C_t \subset \mathbf{P}^3$ be the complete intersection of a surface of degree 5 and a surface of degree t. By the adjunction formula we have $p_a(C_t) = 1 + 5t(t+1)/2$. Hence $p_a(C_t) - p_a(C_{t-1}) = 5t$. This equality explains why we required the inequality $w \le 5t - 1$ in Lemma 4 below and hence in Theorem 4. The restriction $t \ge 15$ in Lemma 4 gives the restriction $g \ge 601$ in Theorem 3.

Lemma 3. *Fix integers z, w with $t \ge 15$ and $0 \le w \le 5t+4$. Let $S \subset \mathbf{P}^3$ be a smooth quintic surface. For a general $A \subset \mathbf{P}^3$ such that $\mathrm{card}(A) = w$ we have $h^1(S, (\mathcal{I}_A)^2(t)) = 0$, $h^0(S, (\mathcal{I}_A)^2(t)) = (5t^2 - 5t + 10)/2 - 3w$.*

Proof. From the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(t-5) \to \mathcal{O}_{\mathbf{P}^3}(t) \to \mathcal{O}_S(t) \to 0\,, \tag{3}$$

we obtain $h^1(S, \mathcal{O}_S(t)) = 0$ and $h^0(S, \mathcal{O}_S(t)) = \binom{t+3}{3} - \binom{t-2}{3} = (5t^2 - 5t + 10)/2$. Thus $h^1(S, (\mathcal{I}_A)^2(t)) = 0$ if and only if $h^0(S, (\mathcal{I}_A)^2(t)) = (5t^2 - 5t + 10)/2 - 3w$. We will use induction on the integer t. We will do only the case $w = 5t + 4$, the general case being similar, but easier: if $w < 5t + 4$ the starting point of the induction is easier while the inductive step is similar to the one we will do; the vanishing of $h^1(S, \mathcal{O}_S(t))$ when $w < 5t + 4$ also follows formally from the corresponding vanishing for $w = 5t+4$. First assume $t = 15$. There are at least two methods to obtain the result. First method; use the same tools as in Severi theory for plane nodal curves as extended in [2] to the case of rational surfaces; here we have $h^1(S, \mathcal{O}_S) = 0$, but since ω_S is positive we cannot obtain for a general $J \subset S$ with $\mathrm{card}(J) = y$ for all integers $y \le h^0(S, \mathcal{O}_S(t))/3$; however, since $h^0(S, \mathcal{O}_S(15)) \gg 237$, we still obtain the starting point of the induction. Second method; we use the inductive approach outlined below, starting from the case $t = 1$ and 3 nodes, but in the range $7 \le t \le 14$ in the step from the integer $t-1$ to the integer t we add in the hyperplane section D 6 points instead of 5 points. Now assume $t \ge 16$ and that the result is true for the integer $t-1$. Let $D \subset S$ be a smooth hyperplane section of D and $E \subset D$ with $\mathrm{card}(E) = 5$. Take a general $B \subset S$ with $\mathrm{card}(B) = 5(t-1)+4$. By the inductive assumption we have $h^1(S, (\mathcal{I}_B)^2(t-1)) = 0$.

Claim. *We have $h^1(S, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(t-1)) = 0$.*

Proof of Claim. Assume

$$h^1(S, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(t-1)) \neq 0.$$

Since $h^1(S, (\mathcal{I}_B)^2(t-1)) = 0$ and E is general in D, the assumption means that 5 general points of D do not give independent conditions to $H^0(S, (\mathcal{I}_B)^2(t-1))$. Hence $h^0(S, (\mathcal{I}_B)^2(t-2)) \ge h^0(S, (\mathcal{I}_B)^2(t-1)) - 4 = (5t^2 - 15t + 12)/2 - 3(5t-6)$.

Take $F \subset B$ with $\mathrm{card}(F) = 5(t-2)+4$. Since B is general in S, if $t \geq 7$ we may apply the inductive assumption for the integer $t-2$ and obtain $h^0(S, (\mathcal{I}_F)^2(t-2)) = (5t^2-25t+40)/2 - 3(5t-11)$. Since B is general, we have $h^0(S, (\mathcal{I}_B)^2(t-2)) \leq \min\{h^0(S, (\mathcal{I}_F)^2(t-2)) - \mathrm{card}(B \backslash F), 0\}$, contradiction; here we use $5t > 29$. In the case $t = 6$ we just make a similar inductive computation, starting with $\mathcal{O}_S(1)$ and then applying the same trick 5 times.

Consider the set of all reducible elements of $\mathbf{P}(H^0(S, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E(t))^*)$ union of a curve in $\mathbf{P}(H^0(S, (\mathcal{I}_B)^2 \otimes \mathcal{I}_E)^*)$ and a plane. Taking a plane containing no point of E and then a plane containing exactly one (resp. two, resp. three) points of E we see that Claim implies $h^1(S, (\mathcal{I}_{B \cup E'})^2(t)) = 0$ for any $E' \subset E$ with $\mathrm{card}(E') = 4$. If the result is not true when we add a fifth point to E', then we abotain two many curves on S singular along their intersection with D, concluding the proof. □

Lemma 4. *Fix integers t, w such that $t \geq 5$ and $0 \leq w \leq 5t-1$. Let $S \subset \mathbf{P}^3$ be a smooth quintic surface with* $\mathrm{Pic}(S) \cong \mathbf{Z}$, *i.e. such that* $\mathrm{Pic}(S)$ *is generated by $\mathcal{O}_S(1)$. For a general $A \subset S$ we have $h^1(S, (\mathcal{I}_A)^2(t)) = 0$, $h^0(S, (\mathcal{I}_A)^2(t)) = (5t^2-5t+10)/2 - 3w$ and there is an integral nodal curve $C \subset S$ such that $A = \mathrm{Sing}(C)$.*

Proof. The first part is a weaker form of Lemma 3. We need to check that the zero-locus of a general $f \in H^0(S, (\mathcal{I}_A)^2(t))$ is an integral nodal curve, C, such that $A = \mathrm{Sing}(C)$. By Lemma 1 we have $h^1(S, (\mathcal{I}_A)^2(t-1)) = 0$. Since A is zero-dimensional, we have $h^2(S, (\mathcal{I}_A)^2(t-2)) = h^2(S, \mathcal{O}_S(t-2))$. By the exact sequence (3) we have $h^2(S, \mathcal{O}_S(t-2)) = 0$ and hence $h^2(S, (\mathcal{I}_A)^2(t-2)) = 0$. By Castelnuovo-Mumford's Lemma ([11], p. 99) the sheaf $(\mathcal{I}_A)^2(t-2)$ is spanned by its global sections. Hence by Bertini's Theorem a general such curve C is nodal and $A = \mathrm{Sing}(C)$. Since by assumption every curve in S is the complete intersection of S with another surface, we easily obtain the irreducibility of C for general C; for reader's sake we will outline a proof. Assume $C = C_1 \cup C_2$ with C_1 (resp. C_2) complete intersection of S with a surface of degree a (resp. $t-a$) for some integer a with $1 \leq a < t$. Hence $A = \mathrm{Sing}(C) = \mathrm{Sing}(C_1) \cup \mathrm{Sing}(C_2) \cup (C_1 \cap C_2)$ and C_1 and C_2 are smooth at each point of $C_1 \cap C_2$ and they intersects transversally. Hence $\mathrm{card}(C_1 \cap C_2) = 5a(t-a)$. Thus either $a = 1$ or $a = t-1$, $w \geq 5t-5$ and C_1 and C_2 are irreducible. Since either C_1 or C_2 is an hyperplane section of S, $w \geq 4 = h^0(S, \mathcal{O}_S(1))$ and A is general, this is impossible. □

Proof of Theorem 4. Let $f : P(A) \to \mathbf{P}^3$ be the blowing-up of A. Set $\mathcal{O}_{P(A)}(1) := f^*(\mathcal{O}_{\mathbf{P}^3}(1))$. Just use $w << (h^0(S, \mathcal{O}_S(t+1))/4$ and the proof of

Theorem 6 to obtain the surjectivity of $\gamma_{\mathcal{O}_{P(A)}(t+1)}$. □

Proof of Theorem 3. There is a unique integer $t \geq 15$ such that $1 + 5(t-1)t/2 < g \leq 1 + 5t(t+1)/2$. Set $w := 1 + 5t(t+1) - g$. Thus $0 \leq w \leq 5t - 1$. Take a smooth quintic surface $S \subset \mathbf{P}^3$ with $\mathrm{Pic}(S) \cong \mathbf{Z}$ and a general $A \subset S$ with $\mathrm{card}(A) = w$. By Lemma 4 there is an integral nodal curve $C \subset S$ such that $A = \mathrm{Sing}(C)$ and C is the complete intersection of S with a degree t surface. Let $f : W \to \mathbf{P}^3$ be the blowing-up of A, $L = f^*(\mathcal{O}_{\mathbf{P}^3}(1))$ and X the strict transform of C in W. Thus X is a smooth and connected genus g curve. By Theorem 4 the Wahl map γ_L is surjective. Thus as in [12], 6.3, to prove the surjectivity of γ_{ω_X} and hence to prove Theorem 3 it is sufficient to show the surjectivity of the restriction map $\beta : H^0(W, \Omega_W(2L)) \to H^0(X, \omega_X^{\otimes 3})$. As in [12], proof of 6.3, the surjectivity of β is easily checked using two exact sequences: first, pass from W to the blowing-up, M, of S along A, and then pass from M to X. □

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