International Journal of Pure and Applied Mathematics
Volume 18 No. 3 2005, 325-329

SECANT SPACES TO PROJECTIVE VARIETIES AND THEIR LIMITS, II

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we study the case $x = n - \dim(X) + 1$ of the following question. For any finite $S \subset \mathbf{P}^n$ let $\langle S \rangle$ denote its linear span. Let $X \subsetneqq \mathbf{P}^n$, $n \geq 2$, be an integral non-degenerate subvariety. For which pairs (Q, x) such that $Q \in \mathbf{P}^n$ and $x \in \mathbb{Z}$, $x \geq 2$, is there $S \subset X$ such that $\sharp(S) = x$ and $Q \in \langle S \rangle$, but $Q \notin \langle S' \rangle$ for every $S' \subsetneqq S$?

AMS Subject Classification: 14N05
Key Words: secant space, osculating space

1. Introduction

In this short note we continue the discussion of the following question raised in [2]. For any finite $S \subset \mathbf{P}^n$ let $\langle S \rangle$ denote its linear span. Let $X \subsetneqq \mathbf{P}^n$, $n \geq 2$, be an integral non-degenerate subvariety.

Question 1. For which pairs (Q, x) such that $Q \in \mathbf{P}^n$ and $x \in \mathbb{Z}$, $x \geq 2$, is there $S \subset X$ such that $\sharp(S) = x$, $Q \in \langle S \rangle$, but $Q \notin \langle S' \rangle$ for every $S' \subsetneqq S$?

In this short note we work over an algebraically closed field $\mathbb{K}$ and prove the following result.

Received: November 11, 2004

Theorem 1. *Let $X \subset \mathbf{P}^n$, $n \geq 2+\dim(X)$, be an integral nondegenerate variety and $Q \in \mathbf{P}^n$. If $\dim(X) > 1$ assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \geq \deg(X) - n + 2$ and that X is neither a cone with vertex containing Q nor contained in a cone with Q in its vertex and with as a basis a minimal degree nondenerate subvariety of $\mathbf{P}^{n-1}$ with dimension $\dim(X)$. There is no $S \subset X$ such that $\sharp(S) = n - \dim(X) + 1$, $Q \in \langle S\rangle$, but $Q \notin \langle S'\rangle$ for every $S' \subsetneqq S$ if and only if $\dim(X) = 1$ and the pair (X, Q) is described either by Example 1 below (any $\mu > 0$ is allowed) or by Example 2 below taking $m = n$ and Y a rational normal curve of $H \cong \mathbf{P}^{n-1}$ (any $e \geq 1$ and any $\mu \geq 0$ are allowed).*

Example 1. Let F_e, $e \geq 0$, the Hirzebruch surface with invariant e, i.e. such that $-e$ is the minimal self-intersection of a section of the ruling of F_e. We fix a ruling $u : F_e \to \mathbf{P}^1$; f is unique if and only if $e > 0$. We have $\mathrm{Pic}(F_e) \cong \mathbf{Z}^{\oplus 2}$ and we take as a basis of $\mathrm{Pic}(F_e)$ a section h of f with $h^2 = -e$ and a class, f, of the ruling u. Thus $h \cdot f = 1$ and $f^2 = 0$. By the projection formula we have $f_*(\mathcal{O}_{F_e}(ah + bf)) \cong \oplus_{i=0}^{a}\mathcal{O}_{\mathbf{P}^1}(b - ie)$ for every $a \geq 0$. Thus $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \geq 0$ and $b < ea$, $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = \sum_{i=1}^{a}(b-ie+1) = (2b+2-ae)(a+1)/2$ if $a \geq 0$ and $b \geq 0$ and $h^1(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \geq 0$ and $b \geq ae-1$. Now fix an integer $n \geq 3$ and set $e := n-1$. Fix an integer $\mu > 0$ and take any integral $T \in |h + (n-1+\mu)f$. Let $\phi : F_{n-1} \to \mathbf{P}^n$ be the morphism induced by $|h + (n-1)f|$. Hence $\phi(F_{n-1})$ is a cone with vertex $Q := \phi(h)$ and a rational normal curve as a basis. The curve $\phi(Y)$ is an integral non-degenerate degree $n-1+\mu$ curve whose linear projection from Q is a rational normal curve. Take any $A \subset \phi(Y)$ such that $Q \notin A$, $\sharp(A) \leq n$. Since no two points of Y are in the same fiber of the ruling of F_{n-1}, we easily see that the linear projection from Q maps A into $\sharp(A)$ linearly independent points. Hence $Q \notin \langle A\rangle$.

Example 2. Fix a prime p, integers $m \geq 3$, $e \geq 1$, $y \geq 1$, $\mu \geq 0$, a hyperplane H of $\mathbf{P}^m$, $Q \in \mathbf{P}^m\backslash H$ and an integral nondegenerate variety $Y \subseteq H$. Assume $\mathrm{char}(\mathbb{K}) = p$. Let $T \subseteq \mathbf{P}^m$ denote the cone with vertex Q and Y as a basis. Choose homogeneous coordinates $x_1, \dots, x_m$ of H and see them as a basis of the m-dimensional linear subspace V of $H^0(Y, \mathcal{O}_Y(1))$ inducing the embedding of Y into H. In the polynomial ring $\mathbb{K}[w, x_1, \dots, x_m]$ in $(m+1)$ variables assign weight p^e to the variable w and weight one to the variables $x_1, \dots, x_m$. Let $A_{y,\mu}$ denote the linear subspace of $\mathbb{K}[w, x_1, \dots, x_m]$ spanned by all monomials with total weight $yp^e+\mu$ and with degree at most y in the variable w. Take a sufficiently general $f \in A_{y,\mu}$ and set $Z := \{P \in T : f(P) = 0\}$. It is easy to check that Z is integral and $\dim(Z) = \dim(Y)$. Take any such integral variety Z. Z has multiplicity μ at Q. Hence $Q \notin Z$ if and only if $\mu = 0$.

The linear projection from Q induces a generically finite surjective morphism $b : Z\backslash\{Q\} \to Y$ with separable degree y, inseparable degree p^e and whose differential is identically zero. Hence every tangent space to Z at one of its smooth points contains Q, i.e. (with the classical terminology) Z is a strange variety and Q is a strange point of Z.

Lemma 1. *Fix integers $n \geq m+2 \geq 3$. Let $C \subset \mathbf{P}^n$, be an integral non-degenerate curve such that for a general $A \subset C$ such that $\sharp(A) = m$ the linear projection of C from $\langle A\rangle$ into $\mathbf{P}^{n-m}$ is a rational normal curve. Then C is a rational normal curve.*

Proof. Since A is general in C and C is non-degenerate, $\dim(\langle A\rangle) = m-1$ and hence the linear projection ϕ from $\langle A\rangle$ maps $\mathbf{P}^n\backslash\langle A\rangle$ into $\mathbf{P}^{n-m}$. By induction on n we reduce to the case $m = 1$. For degree reasons the statement in this particular case is true if and only if the general secant line of C is not a multisecant line of C. This is always true if $\mathrm{char}(\mathbb{K}) = 0$, and in several other cases listed in [4]. Hence we may assume $p := \mathrm{char}(\mathbb{K}) > 0$ and that a general multisecant line D of C satifies $t := \sharp(D \cap C) \geq 3$. Fix a general $P \in C$. Hence $P \notin D$. See the linear projection from C as a linear projection into a general hyperplane H containing D and call $C' \subset H$ the image of C of this projection. By the generality of D we see that a general secant line of C' intersects C' in at least $t \geq 3$ points. A general secant line of a rational normal curve is not a multisecant line, contradiction. □

Lemma 2. *Fix integers $n \geq m+2 \geq 4$. Let $C \subset \mathbf{P}^n$, be an integral non-degenerate curve such that for a general $A \subset C$ such that $\sharp(A) = m$ the linear projection of C from $\langle A\rangle$ into $\mathbf{P}^{n-m}$ is a linearly normal curve with degree $n-m+1$ (and hence with arithmetic genus one). Then $\deg(C) = n+1$ and C is linearly normal (and hence $p_a(C) = 1$).*

Proof. Let $Y \subset \mathbf{P}^3$ be a linearly normal degree 4 integral curve. Hence $p_a(Y) = 1$ (Riemann-Roch). It is well-known and easy to check that Y is the complete intersection of two quadric surfaces. Hence no secant line of Y is a multisecant line. Hence we may copy verbatim the proof of Lemma 1 using Y instead of a plane conic. □

Lemma 3. *Let $C \subset \mathbf{P}^n$, $n \geq 3$, an integral non-degenerate curve such that $p_a(C) = 1$ and $\deg(C) = n+1$, i.e. an integral linearly normal curve with degree $n+1$, and $Q \in \mathbf{P}^n$. There is no $S \subset C$ such that $\sharp(S) = n$, $Q \in \langle S\rangle$ and $Q \notin \langle S'\rangle$ for every $S' \subsetneq S$ if and only if C is singular and Q is the unique* *singular point of C, i.e. if and only if the linear projection of C from Q is a rational normal curve of $\mathbf{P}^{n-1}$.*

Proof. First assume $Q \notin \mathrm{Sing}(C)$ and let H be a general hyperplane containing Q. Then $C \cap H$ is the union of $n+1$ points such that any n of them are linearly independent and span H. If $Q \in C$ take $S := C \cap H \backslash \{Q\}$. If $Q \notin C$, then take as S any n points of $C \cap H$. Now assume $Q \in \mathrm{Sing}(C)$. Since any $n-1$ points of a rational normal curve of $\mathbf{P}^{n-1}$ are linearly, we see that $Q \in A$ for every $A \subset C$ such that $Q \in \langle A \rangle$ and $\sharp(A) \leq n$. □

Proof of Theorem 1. The "if" part was checked in the description of the examples and in the proof of Lemma 3. Hence we will only check the "only if" part. First assume $\dim(X) = 1$. First assume $n = 3$. If X is not contained in a cone with Q as vertex and with a smooth plane conic as a bases, the linear projection $D \subset \mathbf{P}^2$ of C from Q has a two-dimensional family of lines intersecting it in at least 3 points. Hence the plane of $\mathbf{P}^3$ spanned by Q and a general such line is spanned by 3 points different from Q and such that no two of them are collinear with Q. Hence the linear projection of X from Q is a smooth conic and we are in one of the Example 1 and Example 2, for $n = 3$. By Lemma 1 we reduce the case $n > 3$ to the case $n = 3$ taking a linear projection from $n-3$ general points of X; we also used Lemma 2 and Lemma 3 to avoid a small computation. Now assume $\dim(X) \geq 2$ and call $Z \subset \mathbf{P}^{n-1}$ the image of the linear projection of X from Q. We easily check the case in which X is a cone with Q contained in the vertex of X. Hence we may assume $\dim(Z) = \dim(X)$. By the curve case just proved we get that a general codimension $(\dim(X) - 1)$ of Z is a rational normal curve of its linear span. Hence Z is a minimal degree variety, contradicting our assumption. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, On strange projective curves, *Rev. Roum. Math. Pure et Appl.*, **37** (1992), 741-745.

[2] E. Ballico, Secant spaces to projective varieties and their limits, *Preprint*.

[3] V. Bayer, A. Hefez, Strange curves, *Comm. Algebra*, **19** (1991), 3041-3069.

[4] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276**, No. 4 (1987), 565-579.