International Journal of Pure and Applied Mathematics

Volume 18 No. 3 2005, 343-348

ON CERTAIN CLASSES OF P-VALENT FUNCTIONS

Khalida Inayat Noor

Department of Mathematics and Computer Science
College of Science
United Arab Emirates University
P.O. Box 17551, Al-Ain, UNITED ARAB EMIRATES
e-mail: khalidan@uaeu.ac.ae

Abstract: We define certain subclasses of p-valent functions analytic in the unit disk. We study some interesting properties of integral operators:

$$I_n^\alpha(f(z)) = \frac{\alpha+1}{z^{\alpha+1}}\int_0^z t^\alpha(I_{n-1}(f(t)))dt \qquad (\alpha > -1)$$

for $f(z)$ belonging to these classes.

AMS Subject Classification: 30C45, 30C50
Key Words: function with positive real part, p-valent, integral operator, Bazilevic functions

1. Introduction

Let $A(p)$ denote the class of analytic functions

$$f : f(z) = z^p + \sum_{m=p+1}^{\infty} a_m z^m,$$

$p = 1, 2, \ldots$, which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let $P_k(\rho)$ be the class of functions $h(z)$ analytic in E satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} h(z) - \rho}{1-\rho} \right| d\theta \leq k\pi, \tag{1.1}$$

where $z = e^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. This class was introduced in [8]. We note

Received: November 24, 2004

that, for $\rho = 0$, we obtain the class P_k defined by Pinchuk [9] and for $\rho = 0$, $k = 2$, we have the class P of the functions with positive real part. The case $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ. It is known [3] that the class $P_k(\rho)$ is a convex set. Also, for $h \in P_k(\rho)$, we can write

$$h(z) = (\frac{k}{4} + \frac{1}{2})h_1(z) - (\frac{k}{4} - \frac{1}{2})h_2(z), \tag{1.2}$$

where $h_1, h_2 \in P(\rho)$ for $z \in E$.

We introduce the following definition.

Definition 1.1. Let $f \in \mathcal{A}(\rho)$. Then, for $0 \leq \lambda < 1, \mu > 0$, $f \in M_k(p, \lambda, \mu, \rho)$, if it satisfies the condition:

$$\left\{(1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\mu} + \lambda\frac{f'(z)(f(z))^{\mu-1}}{pz^{p\mu-1}}\right\} \in P_k(\rho), \quad \text{for} \quad z \in E. \tag{1.3}$$

Special Cases. (i) $M_2(p, 0, \mu, \rho) = C_p(\mu, \rho) = \{f(z) \in A_p : \text{ Re } \left(\frac{f(z)}{z^p}\right)^{\mu} > \rho,\ 0 \leq \rho < 1,\ z \in E\}$.

(ii) $M_2(p, 1, \mu, \rho) = B_p(\mu, \rho) = \{f(z) \in \mathcal{A}(p) : \text{ Re } \left[\frac{f'(z)(f(z))^{\mu-1}}{pz^{p\mu-1}}\right] > \rho,\ 0 \leq \rho < 1,\ z \in E\}$.

(iii) For $p = 1$, $k = 2$, the class $M_2(1, \lambda, \mu, \rho)$ is a subclass of Bazilevic functions, see [1].

We consider the integral operator $I_n^{\alpha}(f(z))$ defined in [5], as following.

$$I_n^{\alpha}(f(z)) = \frac{\alpha+1}{z^{\alpha+1}} \int_0^z t^{\alpha}(I_{n-1}(f(z)))dt, \quad n \in N,$$

with $I_0^{\alpha}(f(z)) = I_0(f(z)) = (\frac{f(z)}{z^p})^{\mu}$ $(\mu > 0)$. For $\alpha = 0$, the operators I_n^0 were introduced and studied in [7].

2. Preliminaries and Main Results

Lemma 2.1. (see [2]) *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Let $\Psi(u, v)$ be a complex-valued function satisfying the conditions:*

(i) $\Psi(u, v)$ *is continuous in a domain* $\mathbf{D} \subset \mathbf{C^2}$,

(ii) $(1, 0) \in \mathbf{D}$ *and* $\Psi(1, 0) > 0$,

(iii) $\operatorname{Re}\{\Psi(iu_2, v_1)\} \leq 0$, *whenever* $(iu_2, v_1) \in \mathbf{D}$ *and* $v_1 \leq \frac{-1}{2}(1+u_2)$.

If $h(z) = 1+\sum_{m=2}^{\infty} c_m z^m$ *is a function, analytic in* E *such that* $(h(z), zh'(z)) \in \mathbf{D}$ *and* $\operatorname{Re}\{\Psi(h(z),, zh'(z))\} > 0$ *for* $z \in E$, *then* $\operatorname{Re} h(z) > 0$ *in* E.

Lemma 2.2. *Let* $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *be analytic in* E *and let* $\{h(z) + \frac{zh'(z)}{\gamma}\} \in P_k(\rho)$ *for* $\operatorname{Re}\gamma > 0$, $k \geq 2$, $0 \leq \rho < 1$. *Then* $h(z) \in P_k(\rho_0)$, $z \in E$, *where*

$$\rho_0 = \frac{\operatorname{Re}\gamma + 2\rho|r|^2}{\operatorname{Re}\gamma + 2|r|^2}. \tag{2.1}$$

Proof. Let

$$h(z) = (\frac{k}{4} + \frac{1}{2})\,[(1-\rho_0)h_1(z) + \rho_0] - (\frac{k}{4} - \frac{1}{2})\,[(1-\rho_0)h_2(z) + \rho_0]\ ,$$

so $h_i(z)$ is analytic in E, with $h_i(0) = 1$, $i = 1, 2$.

Then

$$\begin{aligned} h(z) + \frac{zh'(z)}{\gamma} &= \{(\frac{k}{4} + \frac{1}{2})[(1-\rho_0)h_1(z) + \frac{(1-\rho_0)zh_1'(z)}{\gamma} + \rho_0] \\ &\quad - (\frac{k}{4} - \frac{1}{2})[(1-\rho_0)h_2(z) + \frac{(1-\rho_0)zh_1'(z)}{\gamma} + \rho_0]\} \in P_k(\rho), \quad z \in E. \end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\{(1-\rho_0)h_i(z) + \frac{(1-\rho_0)}{\gamma}zh_i'(z) + (\rho - \rho_0)\} \in P,$$
$$z \in E,\ i = i, 2.$$

We form the functional $\Psi(u, v)$ by choosing $u = h_i(z), v = zh_i(z)$.

Thus

$$\Psi(u, v) = (1-\rho_0)u + \frac{(1-\rho_0)}{\gamma}v + (\rho - \rho_0).$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the condition (iii) as follows:

$$\begin{aligned} \operatorname{Re}\{\Psi(iu_2, v_1)\} &= (\rho - \rho_0) + \left[\frac{\operatorname{Re}\gamma(1-\rho_0)}{|\gamma|^2}v_1\right] \\ &\leq (\rho - \rho_0) - \left[\frac{\operatorname{Re}\gamma(1-\rho_0)}{2|\gamma|^2}(1+u_2^2)\right] = \frac{A + Bu_2^2}{2C}, \end{aligned}$$

where

$$A = 2|\gamma|^2(\rho_0 - \rho) - \operatorname{Re}\gamma(1-\rho_0), \quad B = -\operatorname{Re}\gamma(1-\rho_0), \quad C = |\gamma|^2 > 0.$$

We notice that $\operatorname{Re}\{\Psi(iu_2, v_1)\} \leq 0$ if and only if $A \leq 0, \quad B \leq 0$ and this gives us ρ_0 as in (2.1) with $0 < \rho_0 < 1$. Therefore, applying Lemma 2.1, $h_i \in P, \quad i = 1, 2.$ and consequently $h \in P_k(\rho_0)$ for $z \in E$. This completes the proof. □

Theorem 2.1. *Let $f \in M_k(p.\lambda, \mu, \rho)$. Then, for $z \in E, \quad I_n^\alpha \in P_k(\rho_n)$, where*

$$\rho_n = \frac{[(1+a)^n - a^n] + a^n \rho_0}{(1+a)^n}, \tag{2.2}$$

with $a = 2(\alpha+1)$ and $\rho_0 = \frac{\lambda+2p\mu\rho}{2p\mu+\lambda}$.

Proof. We shall use the mathematical induction to prove this theorem.

Let

$$H_0(z) = \left(\frac{f(z)}{z^p}\right)^\mu, \text{ with } \mu > 0.$$

By choosing a principle branch of $(\frac{f(z)}{z^p})^\mu$, we note that $H(z)$ is analytic in E with $H_0(0) = 1$. Now

$$(1-\lambda)\left(\frac{f(z)}{z^p}\right)^\mu + \lambda\frac{f'(z)(f(z))^{\mu-1}}{pz^{p\mu-1}} = H_0(z) + \frac{\lambda}{p\mu}zH_0'(z).$$

Since

$$f \in M_k(p, \lambda, \mu, \rho), \quad \left\{H_0(z) + \frac{\lambda}{p\mu}zH_0'(z)\right\} \in P_k(\rho), \quad z \in E$$

and, from Lemma 2.2 with $\gamma = \frac{p\mu}{\lambda}$, it implies that $H \in P_k(\rho_0)$, where $\rho_0 = \frac{\lambda+2p\mu\rho}{\lambda+2p\mu}$.

Thus the result of the theorem is true for $n = 0$.

For $n = 1$, we proceed as follows:

$$H_1(z) = I_1^\alpha(f(z)) = \frac{\alpha+1}{z^{\alpha+}}\int_0^z t^\alpha(I_0^\alpha(f(t)))dt \quad (\alpha > -1)$$

and so

$$\{H_1(z) + \frac{zH_1'(z)}{\alpha+1}\} \in P_k(\rho_0), \quad z \in E.$$

Using Lemma 2.2 with $\gamma = (\alpha + 1)$, we see that

$$H_1(z) = I_1^{\alpha}(f(z)) \in P_k(\rho_1),$$

where $\rho_1 = \frac{1+2(\alpha+1)\rho_0}{1+2(\alpha+1)}$, for $z \in E$. This shows that (2.2) holds true for $n = 1$.

Next we suppose that (2.2) holds true for $n = m$. That is

$$H_m(z) = I_m^{\alpha}(f(z)) \in P_k(\rho_m)$$

and

$$\rho_m = \frac{[(1+a)^m - a^m] + a^m\rho_0}{(1+a)^m}, \quad a = 2(1+\alpha).$$

Proceeding as before we see that

$$\left[H_{m+1}(z) + \frac{zH'_{m+1}(z)}{\alpha+1}\right] = H_m(z) \in P_k(\rho_m), \quad z \in E,$$

where

$$H_{m+1}(z) = I_{m+1}^{\alpha}(f(z)) = \frac{\alpha+1}{z^{\alpha+1}} \int_0^z t^{\alpha}(I_m^{\alpha}(f(t)))dt \quad (\alpha > -1).$$

This implies that $H_{m+1}(z) \in P_k(\rho_{m+1})$, where $\rho_{m+1} = \frac{1+a\rho_m}{(1+a)}$,

$$a = 2(1+\alpha) = \frac{1 + a\left[\frac{\{(1+a)^m - a^m\} + a^m\rho_0}{(1+a)^m}\right]}{(1+a)}$$

$$= \frac{\left[(1+a)^{m+1} - a^{m+1}\right] + a^m\rho_0}{(1+a)^{m+1}},$$

and $\rho_0 = \frac{\lambda+2p\mu\rho}{\lambda+2p\mu}$.

Therefore, we conclude that $I_n^{\alpha} \in P_k)\rho_n)$ for any integer $n \in N_0$. □

As special cases, we note that:

(i) For $f \in C_p(\mu, \rho)$, $\quad \operatorname{Re} I_n^{\alpha}(f(z)) > \rho_n, \quad z \in E$.

(ii) For $f \in B_p(\mu, \rho)$, $\quad \operatorname{Re} I_n^{\alpha}(f(z)) > \rho_n, \quad z \in E$.

(iii) For $f \in M_2(1, 1, 1, \rho)$, $\quad z \in E$, that is,

$$\operatorname{Re} f'(z) > \rho \quad \text{implies} \quad \operatorname{Re}\{\frac{f(z)}{z}\} > \frac{1+2\rho}{3}, \quad z \in E.$$

(iv) With $p = \mu = 1, n = 0, \alpha = 0$, we see that

$$\left[f'(z) + \lambda f''(z)\right] \in P_k(\rho) \text{ implies that } f'(z) \in P_k\left(\frac{\lambda+2\rho}{\lambda+2}\right), \ z \in E.$$

With certain choices of ρ, μ, λ and k, we obtain some partial results discussed in [4], [6], [10].

References

[1] I.E. Bazilevic, On a class of integrability in quadratures of Lowner-Kurfarev equation, *Math. Sb.*, **37** (1955), 471-476.

[2] S.S. Miller, Differential inequalities and Caratheodory functions, *Bull. Amer. Math. Soc.*, **81** (1975), 79-81.

[3] K. Inayat Noor, On subclasses of close-to-convex functions of higher order, *Internat. J. Math. and Math. Sci.*, **15** (1992), 279-290.

[4] M. Obradovic, A property of the class of functions whose derivative has a positive real part, *Publ. Inst. Math.*, **45** (1989), 81-84.

[5] S. Owa, Certain integral operators, *Proc. Japan Acad.*, **67** (1991), 88-93.

[6] S. Owa, M. Nunokuwa, On certain analytic functions and subordination, *Appl. Math. Lett.*, **3** (1990), 41-45.

[7] S. Owa, J. Patel, Properties of certain integral operator, *Southeast Asian Bull. Math.*, **24** (2000), 411-419.

[8] K.S. Padmanabham, R. Parvatham, Properties of a class of functions with bounded boundary rotation, *Ann. Polon. Math.*, **31** (1975), 311-323.

[9] B. Pinchuk, Functions with bounded boundary rotation, *I.J. Math.*, **10** (1971), 7-16.

[10] D.K. Thomas, A note on Bazilevic functions, *Kokyuroku*, **714** (1990), 18-21.