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THE PEDAL CONE SURFACE OF A DEVELOPABLE RULED SURFACE

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Abstract: In this paper, the pedal of a developable ruled surface M in 3-dimensional Euclidean space E^3 is investigated and the components of the position vector of the base curve $\alpha(s)$ are found by means of the pedal of M. Furthermore, the pedal cone surface of the developable ruled surface M is defined and the pedal cone surfaces of the tangent, principal normal and binormal ruled surfaces of the base curve $\alpha(s)$ are investigated.

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1. Introduction

The notion of the pedal of a given surface M in E^3 with respect to a chosen

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origin is well known in literature (see [6, 7, 17, 18, 27]). Georgiou et al [6] have studied the differential geometry of the pedal surface of M with respect to a chosen origin and they investigated the application in geometrical optics. Recently, Kuruoğlu [17] has studied the pedal surface with respect to a point in the interior of a closed, convex and smooth surface in E^3 and given the some new characteristic properties of the pedal surface of M. After, the pedal surface has been generalized by Kuruoğlu and Sarıoğlugil, [18].

On the other hand, ruled surfaces were investigated first by G. Monge who established the partial differential equation satisfied by all ruled surfaces (it is the third order). Thus, ruled surfaces were formed by a one-parameter set of lines and investigated by Hlavaty [10] and Hoschek [12].

Then, the concepts of the striction point, the striction curve and the distribution (Chasles) parameter were earned to the differential geometry by M. Chasles [2]. Furthermore, the theory of ruled surfaces has been applied to kinematics. Especially, the study of one-parameter closed motions became an interesting subject in kinematics after the work of Steiner [28] and Holditch [11]. During the second half of the nineteenth century, many publications about Steiner and Holditch theorems were appeared. For example: Kempe [13, 14, 15, 16], Leudesdorf [21, 22], Elliot [4], Frank [5], Müller [23, 24], Pottman [26], Bottema [1], Thüring [31] and Tölke [32], etc.

Müller showed that the pitch and angle of pitch of closed ruled surfaces in E^3 are simple geometrical integral invariants, [25]. Recently, Several authors have used the these invariants in investigations concerning the generalization of some theorems of the theory of ruled surfaces, for example: Giering [8], Thas [30], Hacısalihoğlu [9], Kuruoğlu and S. Keles [19], etc.

In the present paper, we look for the answer of the question what is the pedal of the developable ruled surface M in E^3 and we show that the pedal of M is a curve. Then, by using the pedal of M, the components of the position vector of the base curve $\alpha(s)$ of M are found. Moreover, the pedal cone surface of the developable ruled surface M is defined and the pedal cone surfaces of the ruled surfaces of the tangent, the principal normal and the binormal of the base curve $\alpha(s)$ of M are investigated.

2. Preliminaries

Let C be a curve in E^3 and O be a fixed point not on C. The locus of the foots of perpendicular drawing from O to the tangents of C with respect to O as origin are called *the pedal curve* of C with respect to O, [29].

Let M be a smooth, convex surface in E^3 and O be a point not on M. If X is the position vector of a point P on M with respect to O as origin and N is the inner unit normal vector of the surface at $P \in M$, then *the support function* h of M is defined by

$$h = - < X \, , \, N > , \tag{2.1}$$

where $<,>$ is the usual metric in E^3.

Geometrically, h is the distance from the origin O to the tangent plane of M at the point of M described by X.

The surface $\bar{M}$ with the position vector

$$\bar{X} = -hN \tag{2.2}$$

of an arbitrary point $\bar{P}$ on the tangent plane $T_M(P)$ of M with respect to O as origin is called *the pedal surface* of M with respect to O.

Geometrically, we can construct the pedal surface $\bar{M}$ as follows:

We draw tangent plane $T_M(P)$ and we get the normal to that plane from O. The normal contacts the plane $T_M(P)$ at a point $\bar{P}$. The locus of all points $\bar{P}$ for all $P \in M$ will give the pedal surface.

We now give a short survey of the most important notions in the theory of ruled surfaces (for details see [3, 20]). A family of one-parameter lines is called *a ruled surface* in 3-dimensional Euclidean space. The equation of the ruled surface then can be written as:

$$M : \varphi(\text{s,v}) = \alpha(\text{s}) + \text{v}\mathbf{e}(\text{s}), \quad \|\mathbf{X}(s)\| = 1, \; s \in I \subset \mathbb{R}, \; v \in \mathbb{R} \, , \tag{2.3}$$

where the curve $\alpha(s)$ is the *base curve*, the lines s =const. *the rulings* of M. If consecutive rulings of a ruled surface intersect, then the surface is said to be *developable*. The analytic condition for this is that

$$< \alpha_s \, , \mathbf{X} \wedge \mathbf{X}_s >= 0. \tag{2.4}$$

3. The Pedal Cone Surface of a Developable Ruled Surface

Let M be a developable ruled surface given by equation (2.3) in E^3. Since the tangent plane is constant along rulings of M, it is clear that the pedal of M is a curve. Thus, for the pedal of M, we can write

$$\bar{\alpha}(s) = \alpha(s) + R(s)\mathbf{T}(s), \quad \|\mathbf{T}(s)\| = \|\alpha_s(s)\| = 1 \, , \tag{3.1}$$

where $R(s)$ is the distance between the points $\alpha(s)$ and $\bar{\alpha}(s)$.

From (3.1) and the definition of pedal of M

$$\bar{\alpha}(s) = < \alpha(s)\,,\, \mathbf{n}(s,v) > \mathbf{n}(s,v)\,,$$

where

$$n(s,v) = \frac{\varphi_s \wedge \varphi_v}{\|\varphi_s \wedge \varphi_v\|}.$$

Then, the pedal $\bar{\alpha}(s)$ of M is obtained as

$$\bar{\alpha}(s) = \alpha_3(s)\mathbf{T} \wedge \mathbf{X} + v\alpha_3(s)\mathbf{X}_s \wedge \mathbf{X}\,, \tag{3.2}$$

where $\alpha_3(s)$ is the third component of the vector $\bar{\alpha}(s)$ with respect to the basis $\{T\ ,\ X\ ,\ \varphi_s \wedge \varphi_v\}$. By the pedal $\bar{\alpha}(s)$ of M, the ruled surface $\bar{M}$ can be defined by

$$\bar{M}: \quad \bar{\varphi}(s\,,\,u) = \bar{\alpha}(s) + u\varphi_s \wedge \varphi_v. \tag{3.3}$$

It is clear that, the rulings of the ruled surface $\bar{M}$ are pass through the fixed point O. Since the base curve $\bar{\alpha}(s)$ of $\bar{M}$ is the pedal of the base curve $\alpha(s)$ of M, $\bar{M}$ is called *the pedal cone surface* of the developable ruled surface M.

From (3.1) and (3.3), we have

$$\bar{\varphi}(s\,,\,u) = (\alpha(s) + R(s)\mathbf{T}(s)) + u\varphi_s \wedge \varphi_v. \tag{3.4}$$

Let $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$ be the components of the position vector $\alpha(s)$ of the base curve of M with respect to the basis $\{\mathbf{T}\ ,\ \mathbf{X}\ ,\ \varphi_s \wedge \varphi_v\}$. By using (3.1) and (3.2), the following relations can be obtained

$$\begin{aligned} \alpha_1(s) &= -R(s)\,, \\ \alpha_2(s) &= 0\,, \\ \alpha_3(s) &= \frac{< \alpha(s)\,,\, \mathbf{T} \wedge \mathbf{X} >}{\sin^2\theta}\,, \end{aligned} \tag{3.5}$$

where θ is the angle between the vectors T and X.

Suppose that, the rulings of the developable ruled surface M are orthogonal to the base curve $\alpha(s)$. Thus, from (2.4) we can write

$$\mathbf{T}(s) = \lambda\,\mathbf{X}(s), \quad \lambda \in IR\,.$$

If $R(s) = 0$ for all $s \in I \subset IR$, the unit normal vector $\bar{n}(s,v)$of the pedal cone surface $\bar{M}$ is

$$\bar{\mathbf{n}}(s,u) = \mathbf{X}(s)\,.$$

Then, we can write the following result.

Corollary 3.1. *Let M be a developable ruled surface such that its rulings are orthogonal to the base curve $\alpha(s)$ and $\bar{M}$ be the pedal cone surface of M. If $R(s) = 0$ for all $s \in I \subset \mathbb{R}$, the unit normal vector $\bar{\mathbf{n}}(s, v)$ of $\bar{M}$ and the rulings of M have the same direction.*

We may occur three ruled surfaces which are obtained by choosing for X in (2.3) one of the three vectors T, N and B of the trihedron of the base curve $\alpha(s)$. This yields the surfaces

$$M_{\mathbf{T}} : \quad \varphi_{\mathbf{T}}(\mathrm{s,v}) = \alpha(\mathrm{s}) + \mathrm{v}\mathbf{T}(\mathrm{s}) \text{ (tangent surface)},$$

$$M_{\mathbf{N}} : \quad \varphi_{\mathbf{N}}(\mathrm{s,v}) = \alpha(\mathrm{s}) + \mathrm{v}\mathbf{N}(\mathrm{s}) \text{ (principal normal surface)},$$

$$M_{\mathbf{B}} : \quad \varphi_{\mathbf{B}}(\mathrm{s,v}) = \alpha(\mathrm{s}) + \mathrm{v}\mathbf{B}(\mathrm{s}) \text{ (binormal surface)}.$$

The tangent surface of the base curve $\alpha(s)$ is a developable surface. If the curve $\alpha(s)$ lies in the plane, then the principal normal surface and the binormal surface of that curve are developable. In this case, for the pedal cone surfaces of the ruled surfaces $M_{\mathbf{T}}$, $M_{\mathbf{N}}$ and $M_{\mathbf{B}}$, we can write

$$\bar{M}_{\mathbf{T}} : \quad \bar{\varphi}_{\mathrm{T}}(\mathrm{s,u}) = \alpha_3(\mathrm{s})B(\mathrm{s}) + \mathrm{u}B(\mathrm{s}),$$

$$\bar{M}_{\mathbf{N}} : \quad \bar{\varphi}_{\mathbf{N}}(\mathrm{s,u}) = \alpha_3(\mathrm{s})B(\mathrm{s}) + \mathrm{u}B(\mathrm{s}),$$

$$\bar{M}_{\mathbf{B}} : \quad \bar{\varphi}_{\mathbf{B}}(\mathrm{s,u}) = \alpha_3(\mathrm{s})N(\mathrm{s}) + \mathrm{u}N(\mathrm{s}),$$

where $\alpha_3(s)$ is the third component of the vector $\alpha(s)$ with respect to the basis $\{T\,,\,N\,,\,B\}$.

Let us consider the pedal cone surface $\bar{M}_{\mathbf{N}}$. Because of $\mathbf{B}_s(s) = -\tau\,(s)\mathbf{N}(s) = 0$, we have $\mathbf{B}(s)$ =constant. On the other hand, by considering the equality $< \alpha_s(s)\,,\,\mathbf{B}(s)\ >= 0$ we get $\alpha_3(s)$ =constant. Now we can give the following result.

Corollary 3.2. (i) *$\bar{M}_{\mathbf{T}}$ whose rulings have the same direction of the binormal vector $\mathbf{B}(s)$ of the base curve $\alpha(s)$ is a cone surface.*

(ii) *$\bar{M}_{\mathbf{N}}$ is consisted of a unique line in the direction of the binormal vector $\mathbf{B}(s)$ of the base curve $\alpha(s)$.*

(iii) *$\bar{M}_{\mathbf{B}}$ whose rulings have the same direction of the principal normal vector $\mathbf{N}(s)$ of the base curve $\alpha(s)$ is a cone surface.*

Example 3.1. The pedal cone surface of the ruled surface $\varphi(s, v) = (s, 0, 0) + v(1, s^2, 0)$ is $\bar{\varphi}(s, u) = u(0, 0, s^2 - 2sv)$. Here, $R(s) = -s$.

Example 3.2. The pedal cone surface of the cylinder surface $\varphi(s, v) = (\cos s, \sin s, 0) + v(0, 0, 1)$ is $\bar{\varphi}(s, u) = (\cos s, \sin s, 0) + u(\cos s, \sin s, 0)$. Here, $R(s) = 0$.

References

[1] O. Bottema, *Ein Problem Der Affinen Kinematik*, Verlag-Oldenburg, München (1956).

[2] M. Chasles, *Corresp. Mathem. et Phys. De Quetelet*, **11** (1839).

[3] M.P. Docarmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs (1976).

[4] E.B. Elliot, Some theorems of kinematics on sphere, *Proc. London Math. Soc.*, **12** (1881).

[5] H. Frank, *Ebene Projektive Kinematik*, Diss. Univ. Karlsruhe (1968).

[6] Chr. Georgiou, Th. Hasanis, D. Koutrofiotis, The pedal of a hypersurface revisited, *Technical Report*, No. 96 (1983).

[7] Chr. Georgiou, Th. Hasanis, D. Koutrofiotis, On the caustic of convex mirror, *Geometriae Dedicata*, **28** (1988), 153-158.

[8] O. Giering, *Vorlesungen Über Höhere Geometrie*, Vieweg, Braunschevag-Wiesbaden (1982).

[9] H.H. Hacısalihoğlu, On the pitch of a closed ruled surface, *Mech. And Mach. Theory*, **7** (1972), 291-305.

[10] V. Hlavaty, *Differentielle Linien Geometrie*, P. Nortdhoff, Groningen (1945).

[11] A. Holditch, *Lady's and Gentleman's Diary for year 1858.*

[12] J. Hoschek, Integral invarianten von regel flachhen, *Arch. Math.*, **XXIV**, 218-224 (1973).

[13] A.B. Kempe, Note on Mr. Leudesdorf's theorem in kinematics, *Messenger Math.*, **7**, 165-167 (1878).

[14] A.B. Kempe, A theorem in kinematics, *Messenger Math.*, **7** (1878).

[15] A.B. Kempe, Proof of the theorem in kinematics, *Messenger Math.*, **8**, No. 42 (1879).

[16] A.B. Kempe, Note on the theorem in kinematics, *Messenger Math.*, **8**, No. 130 (1879).

[17] N. Kuruoğlu, Some new characteristic of the pedal surfaces in Euclidean space E^3, *Pure and Applied Mathematika Sci.*, **XXIII**, No: 1-2 (1986).

[18] N. Kuruoğlu, A. Sarıoğlugil, On the characteristic properties of the hyperpedal surfaces in (n+1)-dimensional Euclidean space E^{n+1}, *Pure and Applied Mathematika Sci.*, **LV**, No: 1-2 (March 2002).

[19] N. Kuruoğlu, S. Keles, Properties of 2-dimensional ruled surfaces in the Euclidean space E^n and Massey's Theorem, *Communication De L'Faculte' des Sciences de L'Universite' d'Ankara*, **32** (1983).

[20] E. Kruppa, *Analytiche und Konstructive Differential Geometria*, Springer, Berlin (1957).

[21] C. Leudesdorf, Theorem in kinematics, *Messenger Math.*, **7** (1877), 125-127.

[22] C. Leudesdorf, Note on the theorem in kinematics, *Messenger Math.*, **8**, 11-12 (1878).

[23] H.R. Müller, Erwesterung des Satzes von Holditch für Geschlossene Raum Kurven, *Abh. Braunschweig Wiss. Ges.*, **31** (1980), 129-135, 1980.

[24] H.R. Müller, Verallgemeinerung Einer Formel Von Steiner, *Abh. Braunschweig Wiss. Ges.*, **31** (1978), 107-113.

[25] H.R. Müller, Uber Geschlossene Bewegungsvorgange, *Monatsh. Math.*, **55** (1951), 209-214.

[26] H. Pottman, Holditch Sicheln, *Arch. Math.*, **44** (1985), 373-378.

[27] G. Salmon, *Analytic Geometry*, Accademic Press., New York (1966).

[28] J. Steiner, *Ges. Werke*, Berlin (1881-1882).

[29] J.D. Struik, *Lectures on Classical Differential Geometry*, Addison-Wesley Press. Inc., Cambridge 42 Mass (1950).

[30] C. Thas, Properties of ruled surfaces in the Euclidean n-space E^n, *Bulletin of the Instute of Math. Acad. Sinica*, **6** (1978).

[31] R. Thüring, Studien über die Holditch'sche Bewegung, *Verh. Naturf. Ges. Basel*, **67** (1956), 575-594.

[32] J. Tölke, Eine affine Veralgemeinerung eines globalen Satsses Vor J. Steiner, Abh, *Braunschweig Wiss. Ges.*, **30** (1979), 1-5.