International Journal of Pure and Applied Mathematics
Volume 19 No. 2 2005, 183-194

A HARDY-HILBERT'S TYPE INTEGRAL INEQUALITY WITH WEIGHTS AND ITS APPLICATION

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Abstract: In this paper, it is shown that a Hardy-Hilbert's type integral inequality with weights can be established by introducing a power-exponent function of the form $x^{1+x}(x \in [0, +\infty))$, and the coefficient $\frac{\pi}{\sin \pi/p}$ is proved to be best possible. In particular, for case $p = 2$, some extensions of the classical Hilbert integral inequality are obtained. As application, some generalizations of Hardy-Littlewood's integral inequality are given.

AMS Subject Classification: 26D15
Key Words: power-exponent function, weight function, Hardy-Hilbert's integral inequality, Hilbert's integral inequality, Hardy-Littlewood's integral inequality

1. Introduction and Lemmas

The famous Hardy-Hilbert's integral inequality is

Received: January 3, 2005
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$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y} dxdy \leq \frac{\pi}{\sin\ \pi/p} \left\{ \int_0^\infty f^p(x)\, dx \right\}^{1/p} \left\{ \int_0^\infty g^q(y)\, dy \right\}^{1/q}, \qquad (1.1)$$

where the coefficient $\frac{\pi}{\sin \pi/p}$ is best possible (see [3]). In particular, when $p = q = 2$, the inequality (1.1) is reduced to the classical Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y} dxdy \ \leq \ \pi \left\{ \int_0^\infty f^2(x)\, dx \right\}^{1/2} \left\{ \int_0^\infty g^2(y)\, dy \right\}^{1/2}, \qquad (1.2)$$

where the coefficient π is best possible.

Recently, the following result was given by introducing power function in the paper [4]:

$$\int_a^b \int_a^b \frac{f(x)\, g(y)}{x^t + y^t} dxdy \leq \left\{ \omega(t, p, q) \int_a^b x^{1-t} f^p(x)\, dx \right\}^{1/p}$$

$$\times \left\{ \omega(t, q, p) \int_a^b x^{1-t} g^q(x)\, dx \right\}^{1/q}, \qquad (1.3)$$

where t is a parameter which is independent of x and y , $\omega(t,\ p,\ q) = \frac{\pi}{t\ \sin\ \pi/pt} - \varphi(q)$ and here function φ is defined by

$$\varphi(r) = \int_0^{a/b} \frac{u^{t-2+1/r}}{1+u^t} du, \quad r = p,\ q.$$

Afterwards, the various extensions on the inequalities (1.1) and (1.2) have appeared in some papers (such as [2], [1], etc.). The purpose of the present paper is that the denominator $x + y$ of the function of the left-hand side of (1.1) is replaced by power-exponent function $x^{1+x} + y^{1+y}$, the new results will be yielded. In particular, for case $p = 2$, the various extensions on (1.2) are obtained. As its application, it is shown that extensions on the Hardy-Littlewood's integral inequality can be established.

For convenience, we need the following lemma.

Lemma 1.1. *Let $h(x) = \frac{1+x}{x} + \ln x$, $x \in (0,\ +\infty)$. Then the minimum of $h(x)$ is 2.*

The proof of Lemma 1.1 is easy, here it is omitted.

2. Theorems and their Corollaries

Define a function by

$$\omega(r,x)=\left(x^{1+x}\left(\frac{1+x}{x}+\ln x\right)\right)^{1-r},\quad x\in(0,+\infty),\tag{2.1}$$

where $r>1$.

Theorem 2.1. *Let* $0<\int_0^\infty \omega(p,x)f^p(x)\,dx<+\infty$, $0<\int_0^\infty \omega(q,x)\times g^q(x)\,dx<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $p\geq q>1$. *Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x^{1+x}+y^{1+y}}dxdy\leq\frac{\pi}{\sin\,\pi/p}\left\{\int_0^\infty \omega(p,x)\,f^p(x)\,dx\right\}^{1/p}\times\left\{\int_0^\infty \omega(q,x)\,g^q(x)\,dx\right\}^{1/q},\tag{2.2}$$

where the weight function $\omega(r,\ x)$ *is defined by (2.1), and the constant factor* $\frac{\pi}{\sin\,\pi/p}$ *is best possible.*

Proof. Let $f(x)=F(x)\left\{\left(x^{1+x}\right)'\right\}^{1/q}$ and $g(y)=G(y)\left\{\left(y^{1+y}\right)'\right\}^{1/p}$. Define two functions by

$$\alpha=\frac{F(x)\left\{(y^{1+y})'\right\}^{1/p}}{(x^{1+x}+y^{1+y})^{1/p}}\left(\frac{x^{1+x}}{y^{1+y}}\right)^{\frac{1}{pq}},\ \beta=\frac{G(y)\left\{(x^{1+x})'\right\}^{1/q}}{(x^{1+x}+y^{1+y})^{1/q}}\left(\frac{y^{1+y}}{x^{1+x}}\right)^{\frac{1}{pq}}.\tag{2.3}$$

Let us apply Hölder's inequality to estimate the right hand side of (2.2) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x^{1+x}+y^{1+y}}dxdy=\int_0^\infty\int_0^\infty \alpha\beta\;dxdy\leq\left\{\int_0^\infty\int_0^\infty \alpha^p dxdy\right\}^{1/p}\left\{\int_0^\infty\int_0^\infty \beta^q dxdy\right\}^{1/q}.\tag{2.4}$$

It is easy to deduce that

$$\int_0^\infty \int_0^\infty \alpha^p dxdy = \int_0^\infty \int_0^\infty \frac{\left(y^{1+y}\right)'}{x^{1+x}+y^{1+y}} \left(\frac{x^{1+x}}{y^{1+y}}\right)^{1/q} F^p(x)\, dxdy$$

$$= \int_0^\infty \omega_q F^p(x)\, dx\,.$$

We compute the weight function ω_q as follows:

$$\omega_q = \int_0^\infty \frac{\left(y^{1+y}\right)'}{x^{1+x}+y^{1+y}} \left(\tfrac{x^{1+x}}{y^{1+y}}\right)^{1/q} dy$$

$$= \int_0^\infty \frac{1}{x^{1+x}+y^{1+y}} \left(\frac{x^{1+x}}{y^{1+y}}\right)^{1/q} d\left(y^{1+y}\right).$$

Let $t = y^{1+y}/x^{1+x}$. Then we have

$$\omega_q = \int_0^\infty \tfrac{1}{1+t}\left(\tfrac{1}{t}\right)^{1/q} dt = \frac{\pi}{\sin\ \pi/q} = \frac{\pi}{\sin \pi/p}\,.$$

Notice that $F(x) = \left\{\left(x^{1+x}\right)'\right\}^{-1/q} f(x)$, hence we have

$$\int_0^\infty \int_0^\infty \alpha^p dxdy = \frac{\pi}{\sin\ \pi/p} \int_0^\infty \left(\left(x^{1+x}\right)'\right)^{1-p} f^p(x)\, dx\,. \tag{2.5}$$

Similarly, we have

$$\int_0^\infty \int_0^\infty \beta^q dxdy = \frac{\pi}{\sin\ \pi/p} \int_0^\infty \left(\left(y^{1+y}\right)'\right)^{1-q} g^q(y)\, dy\,. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x^{1+x}+y^{1+y}} dxdy \ \le \frac{\pi}{\sin \pi/p} \left\{\int_0^\infty \left(\left(x^{1+x}\right)'\right)^{1-p} f^p(x)\, dx\right\}^{1/p}$$

$$\times \left\{\int_0^\infty \left(\left(y^{1+y}\right)'\right)^{1-q} g^q(y)\, dy\right\}^{1/q}. \tag{2.7}$$

We need to show that the constant factor $\frac{\pi}{\sin\,\pi/p}$ contained in (2.7) is best possible.

Define two functions by

$$\tilde{f}\,(x) = \begin{cases} 0\,, & x \in (\,0,\ 1\,)\,, \\ \left(x^{1+x}\right)^{-(1+\,\varepsilon)/p}\left(x^{1+x}\right)'\,, & x \in [\,1,\ +\infty\,)\,, \end{cases}$$

and $\tilde{g}\,(y) = \begin{cases} 0\,, & y \in (\,0,\ 1\,)\,, \\ \left(y^{1+y}\right)^{-(1+\,\varepsilon)/q}\left(y^{1+y}\right)'\,, & y \in [\,1,\ +\infty\,)\,. \end{cases}$

Assume that $0 < \varepsilon < \frac{q}{2p} \quad (p \geq \ q \ > \ 1\,)$, then

$$\int_0^{+\infty}\left(\left(x^{1+x}\right)'\right)^{1-p}\tilde{f}^p\,(x)\,dx = \int_1^{+\infty}\left(x^{1+x}\right)^{-1-\varepsilon}d\left(x^{1+x}\right) = \frac{1}{\varepsilon}\,.$$

Similarly, we have

$$\int_0^{\infty}\left(\left(y^{1+y}\right)'\right)^{1-q}\tilde{g}^{\,q}\,(y)\,dy = \frac{1}{\varepsilon}\,.$$

If $\frac{\pi}{\sin\,\pi/p}$ is not best possible, then there exists $k > 0$ and k less than $\frac{\pi}{\sin\,\pi/p}$ such that

$$\int_0^{\infty}\int_0^{\infty}\frac{\tilde{f}\,(x)\,\tilde{g}\,(y)}{x^{1+x}+y^{1+y}}dxdy < \ k\left(\int_0^{\infty}\left(\left(x^{1+x}\right)'\right)^{1-p}\tilde{f}^p\,(x)\,dx\right)^{1/p}$$
$$\times\left(\int_0^{\infty}\left(\left(y^{1+y}\right)'\right)^{1-q}\tilde{g}^q\,(y)\,dy\right)^{1/q} = \frac{k}{\varepsilon}\,. \qquad (2.8)$$

On the other hand, we have

$$\int_0^{\infty}\int_0^{\infty}\frac{\tilde{f}\,(x)\,\tilde{g}\,(y)}{x^{1+x}+y^{1+y}}dxdy$$
$$= \int_1^{\infty}\int_1^{\infty}\frac{\left\{\left(x^{1+x}\right)^{-(1+\varepsilon)/p}\left(x^{1+x}\right)'\right\}\left\{\left(y^{1+y}\right)^{-(1+\varepsilon)/q}\left(y^{1+y}\right)'\right\}}{x^{1+x} \quad + \quad y^{1+y}}dxdy$$

$$= \int_1^{\infty} \left\{ \int_1^{\infty} \frac{\left(y^{1+y}\right)^{-(1+\varepsilon)/q}}{x^{1+x} + y^{1+y}} d\left(y^{1+y}\right) \right\} \left\{ \left(x^{1+x}\right)^{-(1+\varepsilon)/p} \left(x^{1+x}\right)' \right\} dx$$

$$= \int_1^{\infty} \left\{ \int_{1/x^{1+x}}^{\infty} \frac{1}{1+t} \left(\frac{1}{t}\right)^{-(1+\varepsilon)/q} dt \right\} \left(x^{1+x}\right)^{-1-\varepsilon} d\left(x^{1+x}\right)$$

$$= \frac{1}{\varepsilon} \int_{x^{-(1+x)}}^{\infty} \frac{1}{1+t} \left(\frac{1}{t}\right)^{-(1+\varepsilon)/q} dt\,.$$

If the lower limit $x^{-(1+x)}$ of the integral is replaced by zero, then the resulting error is smaller than $\frac{\left(x^{-(1+x)}\right)^{\alpha}}{\alpha}$, where α is positive and α is independent of ε. In fact, we have

$$\int_0^{x^{-(1+x)}} \frac{1}{1+t} \left(\frac{1}{t}\right)^{(1+\varepsilon)/q} dt \; < \; \int_0^{x^{-(1+x)}} t^{-(1+\varepsilon)/q} dt = \frac{\left(1/x^{1+x}\right)^{\beta}}{\beta},$$

where $\beta = 1 - (1+\varepsilon)/q$. If $0 < \varepsilon < \frac{q}{2p}$, then we may take α such that

$$\alpha = 1 - \frac{1 + q/2p}{q} = \frac{1}{2p}.$$

Consequently, we get

$$\int_0^{\infty} \int_0^{\infty} \frac{\tilde{f}(x)\, \tilde{g}(y)}{x^{1+x} + y^{1+y}} dx dy > \frac{1}{\varepsilon} \left\{ \frac{\pi}{\sin \pi/p} + \circ(1) \right\} \quad (\varepsilon \to 0\,). \tag{2.9}$$

Clearly, when ε is small enough, the inequality (2.8) is in contradiction with (2.9). Therefore, $\frac{\pi}{\sin \pi/p}$ is the best possible value of which the inequality (2.7) keeps valid.

Let $u = x^{1+x}$ and $v = y^{1+y}$. Then $u' = x^{1+x}(\frac{1+x}{x} + \ln x)$ and $v' = y^{1+y}(\frac{1+y}{y} + \ln y)$. Substituting them into (2.7), the inequality (2.2) yields at once. The proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let*

$$0 < \int_0^{\infty} x^{(1+x)(1-p)} f^p(x)\, dx < +\infty,$$

$$0 < \int_0^\infty x^{(1+x)(1-q)} g^q(x)\,dx < +\infty,$$

$\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \geq q > 1$. *Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x^{1+x} + y^{1+y}} dxdy \leq \frac{\pi}{2\sin\ \pi/p} \left\{ \int_0^\infty x^{(1+x)(1-p)} f^p(x)\,dx \right\}^{1/p} \times \left\{ \int_0^\infty x^{(1+x)(1-q)} g^q(x)\,dx \right\}^{1/q}. \tag{2.10}$$

Proof. Let $u(x) = x^{1+x}$ and $v(y) = y^{1+y}$. Then $u'(x) = u(x)\,h(x)$ and $v'(y) = v(y)\,h(y)$. By Lemma 1.1, $\min\{h(x)\} = \min\{h(y)\} = 2$. Hence we have $\max\left\{\frac{1}{h(x)}\right\} = \max\left\{\frac{1}{h(y)}\right\} = \frac{1}{2}$. Notice that $p \geq q > 1$, consequently the inequality (2.7) can be reduced to the following form:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x^{1+x} + y^{1+y}} dxdy \\ &\leq \frac{\pi}{\sin \pi/p} \left\{ \int_0^\infty \left(u'(x)\right)^{1-p} f^p(x)\,dx \right\}^{1/p} \left\{ \int_0^\infty \left(v'(y)\right)^{1-q} g^q(y)\,dy \right\}^{1/q} \\ &\leq \frac{\pi}{\sin\ \pi/p} \left\{ \int_0^\infty \left(\tfrac{1}{2}\right)^{p-1} x^{(1+x)(1-p)} f^p(x)\,dx \right\}^{1/p} \\ &\quad \times \left\{ \int_0^\infty \left(\tfrac{1}{2}\right)^{q-1} y^{(1+y)(1-q)} g^q(y)\,dy \right\}^{1/q}. \end{aligned} \tag{2.11}$$

The inequality (2.10) follows from (2.11) after simplifications. The Theorem 2.2 is proved. □

In particular, for case $p = 2$, some extensions of (1.2) are obtained. According to Theorem 2.1, we get the following results.

Corollary 2.3. *Define a function by*

$$\omega(x) = x^{1+x} \left(\frac{1+x}{x} + \ln x \right), \quad x \in (0,\ +\infty).$$

If $0<\int_0^\infty \omega^{-1}(x)\, f^2(x)\, dx<+\infty$ *and* $0<\int_0^\infty \omega^{-1}(x)\, g^2(x)\, dx<+\infty$, *then*

$$\int_0^\infty\int_0^\infty \frac{f(x)\, g(y)}{x^{1+x}+y^{1+y}}dxdy \le \pi\left\{\int_0^\infty \omega^{-1}(x)\, f^2(x)\, dx\right\}^{1/2} \times\left\{\int_0^\infty \omega^{-1}(x)\, g^2(x)\, dx\right\}^{1/2}, \quad (2.12)$$

where the constant factor π *is best possible.*

Corollary 2.4. *If* $0<\int_0^\infty x^{-(1+x)}\left(\frac{1+x}{x}+\ln x\right)^{-1} f^2(x)\, dx<+\infty$, *then*

$$\int_0^\infty\int_0^\infty \frac{f(x)\, f(y)}{x^{1+x}+y^{1+y}}dxdy \le \pi\int_0^\infty x^{-(1+x)}\left(\frac{1+x}{x}+\ln x\right)^{-1} f^2(x)\, dx\,, \quad (2.13)$$

where the constant factor π *is best possible.*

Basing on Theorem 2.2, the following inequalities are established.

Corollary 2.5. *Let* $0<\int_0^\infty x^{-(1+x)}f^2(x)\, dx<+\infty, 0<\int_0^\infty x^{-(1+x)}$ $\times g^2(x)\, dx<+\infty$, *Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)\, g(y)}{x^{1+x}+y^{1+y}}dxdy \le \frac{\pi}{2}\left\{\int_0^\infty x^{-(1+x)}f^2(x)\, dx\right\}^{1/2} \times\left\{\int_0^\infty x^{-(1+x)}g^2(x)\, dx\right\}^{1/2}. \quad (2.14)$$

Corollary 2.6. *If* $0<\int_0^\infty x^{-(1+x)}f^2(x)\, dx<+\infty$, *then*

$$\int_0^\infty\int_0^\infty \frac{f(x)\, f(y)}{x^{1+x}+y^{1+y}}dxdy \le \frac{\pi}{2}\int_0^\infty x^{-(1+x)}f^2(x)\, dx. \quad (2.15)$$

3. Application

In this section, we will give various extensions on Hardy-Littlewood's integral inequality.

Let $f(x) \in L^2(0,\ 1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)\, dx, \quad n = 0,\ 1,\ 2, \cdots,$$

then we have the Hardy-Littlewood's inequality (see [3]) of the form

$$\sum_{n=0}^{\infty} a_n^2 \ < \ \pi \int_0^1 f^2(x)\, dx\,, \tag{3.1}$$

where π is the best constant that keeps (3.1) valid. In our previous paper [5], the inequality (3.1) was extended and established the following inequality:

$$\int_0^{\infty} f^2(x)\, dx \ < \ \pi \int_0^1 h^2(x)\, dx\,, \tag{3.2}$$

where $f(x) = \int_0^1 t^x\, h(x)\, dx$, $x \in [0,\ +\infty)$.

Afterwards the inequality (3.2) was refined into form in the paper [6]:

$$\int_0^{\infty} f^2(x)\, dx \leq \pi \int_0^1 t\, h^2(t)\, dt. \tag{3.3}$$

We will further extend the inequality (3.3), some new results will be obtained.

Theorem 3.1. *Let* $h(t) \in L^2[0,\ 1]$, $h(t) \neq 0$. *Define a function by*

$$f(x) = \int_0^1 t^{u(x)}\, |h(t)|\, dt\,, \quad x \in [0,\ +\infty)\,,$$

where $u(x) = x^{1+x}$. *If* $0 < \int_0^{\infty} \omega(r,\ x) f^r(x)\, dx < +\infty$, *where the weight function* $\omega(r,\ x)$ *is defined by (2.1),* $(r = p,\ q\ )$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \geq q > 1$, *then*

$$\left(\int_0^{\infty} f^2(x)\, dx \right)^2 \ < \ \frac{\pi}{\sin\ \pi/p} \left(\int_0^{\infty} \omega(p,\ x)\, f^p(x)\, dx \right)^{1/p}$$

$$\times \left(\int_0^\infty \omega(q,\ y)\, f^q(y)\, dy \right)^{1/q} \int_0^1 t\, h^2(t)\, dt\,, \tag{3.4}$$

where the constant factor $\frac{\pi}{\sin\ \pi/p}$ *in (3.4) is best possible.*

Proof. Let us write $f^2(x)$ in form:

$$f^2(x) = \int_0^1 f(x)\, t^{u(x)}\, |h(t)|\, dt.$$

We apply, in turn, Schwarz's inequality and Theorem 2.1 we have

$$\left(\int_0^\infty f^2(x)\, dx \right)^2 = \left\{ \int_0^\infty \left(\int_0^1 f(x)\, t^{u(x)}\, |h(t)|\, dt \right) dx \right\}^2$$

$$= \left\{ \int_0^1 \left(\int_0^\infty f(x)\, t^{u(x)-1/2} dx \right) t^{1/2}\, |h(t)|\, dt \right\}^2$$

$$\leq \int_0^1 \left(\int_0^\infty f(x)\, t^{u(x)-1/2} dx \right)^2 dt \int_0^1 t\, h^2(t)\, dt$$

$$= \int_0^1 \left(\int_0^\infty f(x)\, t^{u(x)-1/2} dx \right) \left(\int_0^\infty f(y)\, t^{u(y)-1/2} dy \right) dt \int_0^1 t\, h^2(t)\, dt$$

$$= \int_0^1 \left(\int_0^\infty \int_0^\infty f(x)\, f(y)\, t^{u(x)+u(y)-1} dx dy \right) dt \int_0^1 t\, h^2(t)\, dt$$

$$= \left(\int_0^\infty \int_0^\infty \frac{f(x)\, f(y)}{u(x)\ +\ u(y)} dx dy \right) \int_0^1 t\, h^2(t)\, dt$$

$$\leq \frac{\pi}{\sin \pi/p} \left\{ \int_0^\infty \omega(p,\ x)\, f^p(x)\, dx \right\}^{1/p} \left\{ \int_0^\infty \omega(q,\ y)\, f^q(y)\, dy \right\}^{1/q}$$

$$\times \int_0^1 t\, h^2(t)\, dt. \tag{3.5}$$

Since $h(t) \neq 0$, $f^2(x) \neq 0$. It is impossible to take equality in (3.5). We therefore complete the proof of the theorem. □

Using Theorem 3.1 and Theorem 2.2, the following results are attained.

Corollary 3.2. *The functions* $h(t)$, $f(x)$ *and* $u(x)$ *with the assumptions as Theorem 3.1, if* $0 < \int_0^\infty x^{(1+x)(1-r)} f^r(x)\,dx < +\infty$ $(r = p, q)$, *then*

$$\left(\int_0^\infty f^2(x)\,dx\right)^2 < \frac{\pi}{2\sin\ \pi/p}\left(\int_0^\infty x^{(1+x)(1-p)} f^p(x)\,dx\right)^{1/p} \times \left(\int_0^\infty y^{(1+y)(1-q)} f^q(y)\,dy\right)^{1/q} \int_0^1 t\,h^2(t)\,dt. \qquad (3.6)$$

In particular, when $p = q = 2$, we have the following result.

Corollary 3.3. *The functions* $h(t)$, $f(x)$*and* $u(x)$*with the assumptions as Theorem 3.1, if* $0 < \int_0^\infty x^{-(1+x)}\left(\frac{1+x}{x} + \ln x\right)^{-1} f^2(x)\,dx < +\infty$, *then*

$$\left(\int_0^\infty f^2(x)\,dx\right)^2 < \pi\left(\int_0^\infty x^{-(1+x)}\left(\tfrac{1+x}{x} + \ln x\right)^{-1} f^2(x)\,dx\right)\int_0^1 t\,h^2(t)\,dt\,, \qquad (3.7)$$

where the constant factor π *in (3.7) is best possible.*

The inequalities (3.4), (3.6) and (3.7) are extensions of (3.3).

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