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HILBERT MODULES

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Abstract: We shall extend the notion of Hilbert rings to Hilbert modules. We call an R-module M a Hilbert module if every prime submodule of M is the intersection of all the maximal submodules containing it. It will be shown that every finitely generated module over a ring R is a Hilbert module if and only if R is a Hilbert ring. Moreover, we prove that a finitely generated R-module M is a Hilbert module if and only if $R/\mathrm{Ann}_R(M)$ is a Hilbert ring.

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1. Introduction

Throughout this paper, R denotes a non-zero commutative ring with identity and every module is unital. A ring is local if it has exactly one maximal ideal. By a semilocal ring we shall mean a ring which has only finitely many maximal ideals. We shall denote the Krull dimension of R by $\dim R$. Recall that a Hilbert ring (HR for short) is a ring R, having the property that every prime ideal in R is the intersection of all the maximal ideals containing it. Obviously, $\mathbb{Z}$ is a Hilbert ring and if F is a field, both F and $F[X]$ are Hilbert rings. On the other hand, if R is a local ring of Krull dimension at least 1, then R is never

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a Hilbert ring.

We list some facts about Hilbert rings (see, for example, [8]):

(i) Every homomorphic image of a HR is also a Hilbert ring.

(ii) Any ring of Krull dimension 0 is a Hilbert ring.

(iii) R is a Hilbert ring if and only if $R[X_1, \ldots, X_n]$ is too.

(iv) Let $R \subseteq S$ be rings, S integral over R. Then S is a HR if and only if R is too.

(v) Every finitely generated ring over a HR is a Hilbert ring.

(vi) A finite direct sum of rings is a Hilbert ring if and only if each summand is too.

(vii) Every non-semilocal Dedekind domain is a Hilbert ring.

Let M be an R-module. For any submodule K of M, let $(K : M)$ denote the set of elements r in R such that $rM \subseteq K$. Note that $(K : M)$ is the annihilator of the R-module M/K, and hence $(K : M)$ is an ideal of R. A proper submodule K of M is said to be prime (or P-prime) if $rm \in K$ for $r \in R$ and $m \in M$ implies that either $m \in K$ or $r \in P = (K : M)$. Clearly if K is a prime submodule of M, then $(K : M)$ is a prime ideal of R (see [5]). Various authors have generalized the theory of prime ideals of R to prime submodules of M. For example, a generalization of the Principal Ideal Theorem is given for modules [4]. In [6, 7] the concept of catenary rings have been generalized to modules and catenary modules have been characterized. In this paper we shall extend the notion of Hilbert rings to modules and we shall prove some related results.

2. Hilbert Modules

Definition. An R-module M is said to be a Hilbert R-module or simply a Hilbert module (HM for short) if every prime submodule in M is the intersection of all the maximal submodules containing it.

Examples 2.1. (i) Any Hilbert ring R is a Hilbert R-module.

(ii) Any vector space V over a field F is a HM. For, let W be a prime subspace of V and Y a basis of W. Then there exists a basis X of V containing Y. Let W_x be the subspace of V generated by the set $X \backslash \{x\}$, for every $x \in X \backslash Y$. Then each W_x is maximal in V and $W = \bigcap_{x \in X \backslash Y} W_x$.

(iii) Any finitely generated divisible module over an integral domain is a HM (see [7, Proposition 1.16] and part (ii) above).

One can show the following lemma.

Lemma 2.2. *Any homomorphic image of a HM is a HM.*

Corollary 2.3. *Every cyclic R-module is a HM if and only if R is a HR.*

Proof. ($\Longrightarrow$). It is trivial.

($\Longleftarrow$). Every cyclic R-module is isomorphic to an R-module of the form R/I, where I is an ideal of R. Now Lemma 2.2 completes the proof. □

Examples 2.4. (i) If R is not a HR, then no non-zero free R-module is a HM (see Lemma 2.2).

(ii) Q is not a Hilbert $\mathbb{Z}$-module. In general let R be an integral domain and K the quotient field of R. If $K \neq R$, then K is not a Hilbert R-module. Since the zero R-submodule of K is prime, but K doesn't have any maximal R-submodule. For, let M be a maximal R-submodule of K. By [7, Lemma 1.12], $(M : K) = 0$ which is a maximal ideal of R. Thus $K = R$, a contradiction.

Remark. Let M be an R-module and I an ideal of R such that $I \subseteq \operatorname{Ann}_R(M)$, so that, M can be regarded as an (R/I)-module in a natural way. Recall also that a subset of M is an R-submodule if and only if it is an (R/I)-submodule. Moreover, the prime (resp. maximal) submodules of the R-module M and the prime (resp. maximal) submodules of the (R/I)-module M are the same. Therefore, it follows that M is a Hilbert R-module if and only if M is a Hilbert (R/I)-module. Moreover, $J_R(M) = J_{R/I}(M)$, where $J_R(M)$ denotes the Jacobson radical of the R-module M.

Example 2.5. Let $\mathcal{M}$ be a maximal ideal of R. Then $\mathcal{M}/\mathcal{M}^2$ is a Hilbert $(R/\mathcal{M})$-module and so $\mathcal{M}/\mathcal{M}^2$ is a Hilbert R-module.

Lemma 2.6. *Let M be an R-module. Then the following are equivalent:*

(i) M is a Hilbert R-module.

(ii) M/K is a Hilbert (R/P)-module, for each prime submodule K of M with $P = (K : M)$.

(iii) $J_R(M/K) = 0$, for each prime submodule K of M.

Proof. (i)$\Longrightarrow$(ii). Let K be a prime submodule of M with $P = (K : M)$. Then by Lemma 2.2, M/K is a Hilbert R-module. Since $P = \operatorname{Ann}_R(M/K)$, the above remark completes the proof.

(ii)$\Longrightarrow$(iii). Let K be a P-prime submodule of M. Then the zero submodule of the (R/P)-module M/K is prime. By hypothesis we have $J_{R/P}(M/K) = 0$. On the other hand, $J_R(M/K) = J_{R/P}(M/K)$. Therefore, $J_R(M/K) = 0$.

(iii)$\Longrightarrow$(i). It is clear. □

Corollary 2.7. *If $\dim R = 0$, then every R-module is a HM.*

Proof. It follows from Lemma 2.6(ii) and Examples 2.1(ii). □

The above corollary gives that every module over a semilocal ring R is a HM if and only if $\dim R = 0$.

Note. For the remainder of this section we assume that all modules are finitely generated.

The following lemma is useful.

Lemma 2.8. *Let R be an integral domain with zero Jacobson radical. Then every torsion-free R-module has zero Jacobson radical.*

Proof. Let M be a non-zero R-module which is torsion-free. There exists a free submodule F of M such that M/F is a torsion module (F is generated by a maximal independent subset of M). Because M/F is finitely generated, there exists a non-zero element r in R such that $r(M/F) = 0$, i.e., $rM \subseteq F$. Note that M is isomorphic to rM which is an R-submodule of F. By [2, Proposition 9.14 and Proposition 9.19], rM (and hence M) has zero Jacobson radical. □

Proposition 2.9. *Every R-module is a HM if and only if R is a HR.*

Proof. ($\Longrightarrow$). It is clear.

($\Longleftarrow$). Let M be an R-module and K a prime submodule of M with $P = (K : M)$. Then the (R/P)-module M/K is finitely generated and torsion free. Note that R/P is an integral domain with zero Jacobson radical. By Lemma 2.8, $J_R(M/K) = J_{R/P}(M/K) = 0$. It follows that M is a HM, by Lemma 2.6. □

The above proposition shows that every module over a non-semilocal Dedekind domain is a HM.

Remark. Let P be a prime ideal of R. Using Lemma 2.8, we have P is an intersection of maximal ideals of R if and only if every P-prime submodule of every R-module M is an intersection of maximal submodules of M.

Proposition 2.10. *Let M be a faithful R-module. Then M is a HM if and only if R is a HR.*

Proof. ($\Longrightarrow$). Let P be a prime ideal of R. By [1, Theorem 1.3], there exists a P-prime submodule K of M. Therefore we have $K = \bigcap_{i\in I} M_i$ with each M_i maximal in M and each $\mathcal{M}_i = (M_i : M)$ maximal in R. Thus $P = (K : M) = (\bigcap_{i\in I} M_i : M) = \bigcap_{i\in I}(M_i : M) = \bigcap_{i\in I} \mathcal{M}_i$. Therefore R is a HR as required.

($\Longleftarrow$). It follows by Proposition 2.9. □

Being faithful in Proposition 2.10 is necessary. Every simple module over a discrete valuation ring provides an easy counterexample.

Corollary 2.11. *An R-module M is a HM if and only if $R/\operatorname{Ann}_R(M)$ is a HR.*

Proof. Let M be a non-zero R-module. Then M is a Hilbert R-module if and only if M is a Hilbert $(R/\mathrm{Ann}_R(M))$-module which is equivalent to $R/\mathrm{Ann}_R(M)$ is a HR, by Proposition 2.10. □

Suppose that R is a semilocal Dedekind domain which is not a field. Corollary 2.11 shows that an R-module M is a HM if and only if $\mathrm{Ann}_R(M) \neq 0$ which is equivalent to M is torsion.

As we know if R is an Artinian ring, then R is a HR. We now extend this result for modules.

Corollary 2.12. *Every Artinian R-module M is a HM. In particular, if $l_R(M) < \infty$, then M is a HM.*

Proof. Let M be a non-zero Artinian R-module. Then $R/\mathrm{Ann}_R(M)$ is an Artinian ring and hence a Hilbert ring. By Corollary 2.11, M is a HM. □

Proposition 2.13. *Let $M_1, \dots, M_n$ be R-modules. Then $\bigoplus_{i=1}^{n} M_i$ is a HM if and only if each M_i is a HM.*

Proof. ($\Longrightarrow$). It follows by Lemma 2.2.

($\Longleftarrow$). Clearly, we can assume that M_i is a non-zero HM for $i = 1, \dots, n$. By Corollary 2.11, each $R/\mathrm{Ann}_R(M_i)$ is a HR. Thus

$$R/(\bigcap_{i=1}^{n} \mathrm{Ann}_R(M_i))$$

is a HR. Since $\mathrm{Ann}_R(\bigoplus_{i=1}^{n} M_i) = \bigcap_{i=1}^{n} \mathrm{Ann}_R(M_i)$, $\bigoplus_{i=1}^{n} M_i$ is a Hilbert R-module, by Corollary 2.11. □

Proposition 2.14. *Every finitely generated submodule of a HM is a HM.*

Proof. The result follows by Corollary 2.11. □

We know that R is a HR if and only if $R[X_1, \dots, X_n]$ is a HR. Now we prove a similar result for modules.

Proposition 2.15. *An R-module M is a HM if and only if $M[X]$ is a Hilbert $R[X]$-module.*

Proof. Let M be a non-zero R-module. By Corollary 2.11, M is a HM if and only if $R/\mathrm{Ann}_R(M)$ is a HR. On the other hand, $R/\mathrm{Ann}_R(M)$ is a HR if and only if $(R/\mathrm{Ann}_R(M))[X]$ is a HR. Since the rings $(R/\mathrm{Ann}_R(M))[X]$ and $R[X]/\mathrm{Ann}_{R[X]}M[X]$ are isomorphic, M is a Hilbert R-module if and only if $R[X]/\mathrm{Ann}_{R[X]}M[X]$ is a HR which is equivalent to $M[X]$ is a Hilbert $R[X]$-module, by Corollary 2.11. □

Proposition 2.16. *An R-module M is a HM if and only if $M[X_1, \ldots, X_n]$ is a Hilbert $R[X_1, \ldots, X_n]$-module.*

Proof. Similar to the proof of Proposition 2.15. □

For the next lemma we assume that M is an R-module which is not necessarily finitely generated and $\mathcal{N}$ is the nilradical of R.

Lemma 2.17. *The following are equivalent:*

(i) M is a Hilbert R-module.

(ii) $M/\mathcal{N}M$ is a Hilbert R-module.

(iii) $M/\mathcal{N}M$ is a Hilbert $(R/\mathcal{N})$-module.

Proof. (i)$\Longrightarrow$(ii). It follows by Lemma 2.2.

(ii)$\Longrightarrow$(iii). It is clear by the first remark.

(iii)$\Longrightarrow$(i). If K is a P-prime submodule of the R-module M, then $PM \subseteq K$ and so $\mathcal{N}M \subseteq K$. Now it is clear that $K/\mathcal{N}M$ is a $(P/\mathcal{N})$-prime submodule of the $(R/\mathcal{N})$-module $M/\mathcal{N}M$. By hypothesis we have $K/\mathcal{N}M = \bigcap_{i\in I}(M_i/\mathcal{N}M)$ with each $M_i/\mathcal{N}M$ maximal in $M/\mathcal{N}M$. Hence $K = \bigcap_{i\in I} M_i$, where the M_i are maximal in M. □

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References

[1] S. Abu-Saymeh, On dimensions of finitely generated modules, *Comm. Algebra*, **23**, No. 3 (1995), 1131-1144.

[2] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York (1974).

[3] Z.A. El-Bast, P.F. Smith, Multiplication modules, *Comm. Algebra*, **16**, No. 4 (1988), 755-779.

[4] S.M. George, R.L. McCasland, P.F. Smith, A principal ideal theorem analogue for modules over commutative rings, *Comm. Algebra*, **22** (1994), 2083-2099.

[5] C.P. Lu, Prime submodules of modules, *Comment. Math. Univ. Sancti Pauli*, **33**, No. 1 (1984), 61-69.

[6] S. Namazi, H. Sharif, Catenary modules, *Acta Math. Hungar.*, **85**, No. 3 (1999), 211-218.

[7] S. Namazi, H. Sharif, Catenary modules II, *Honam Math. J.*, **22**, No. 3 (2000), , 9-16.

[8] D.G. Northcott, *Lessons on Rings*, Modules and Multiplicities, Cambridge University Press (1968).