International Journal of Pure and Applied Mathematics
Volume 20 No. 1 2005, 81-95

ON A NONLINEAR COUPLED SYSTEM

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Abstract: We prove the existence and uniqueness of the weak solutions of the Cauchy problem for the system

$$u'' - \Delta u + f(u, v)u = h_1,$$
$$v'' - \Delta v + g(u, v)v = h_2,$$

assuming only that the functions f and g are continuous in the first variable and Lipschitz in the second one. In one dimensional case, this non-linear system describes the motion of charged meson in an electromagnetic field. In previous investigations, this system has been studied supposing the functions f and g more regular then in our case. Therefore, we improve the earlier results, as we can see in Introduction, by weakening the regularity assumptions on the functions f and g.

AMS Subject Classification: 35F25, 35L70
Key Words: Cauchy problem, existence, uniqueness, weak local solutions

Received: March 5, 2005

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1. Introduction

Let Ω be a open bounded set in $\mathbb{R}^n$ with smooth boundary and Q the cylinder $\Omega\times]0,T[$ for $T>0$ a real number.

We are concerned with the existence and uniqueness of local weak solutions for the Cauchy problem

$$\begin{aligned} u''(x,t)-\Delta u(x,t)+f\big(u(x,t),v(x,t)\big)u(x,t)=h_1(x,t) \text{ in } Q,\\ v''(x,t)-\Delta v(x,t)+g\big(u(x,t),v(x,t)\big)v(x,t)=h_2(x,t) \text{ in } Q, \end{aligned} \tag{1.1}$$

$$\begin{aligned} u(x,0)=u_0(x) \text{ and } v(x,0)=v_0(x) \text{ with } x\in\Omega,\\ u'(x,0)=u_1(x) \text{ and } v'(x,0)=v_1(x) \text{ with } x\in\Omega, \end{aligned} \tag{1.2}$$

where $''$ denotes $\frac{\partial^2}{\partial t^2}$, $\Delta u(x,t)$ is the usual Laplace operator in $\mathbb{R}^n$ of the function $u(x,t)$ and f, g, h_1 and h_2 are real functions with real values. The hypotheses about these functions will be fixed in the Section 2.

System (1.1) is a mathematical generalization of the following model idealized by Segal [11]

$$\begin{aligned} \mathcal{A}u+\alpha^2u+\gamma^2v^2u=0,\\ \mathcal{A}v+\beta^2v+\sigma^2u^2v=0, \end{aligned} \tag{1.3}$$

where $\mathcal{A}$ is the d'Alambertian operator given by $\frac{\partial^2}{\partial t^2}+\Delta$. System (1.3) describes the interaction of scalar fields u and v of mass α and β, respectively, with interaction constants γ and σ. Such system defines the motion of charged meson in an electromagnetic field.

To be clear the objectives of the present work let us give an historical summary of problems associated with system (1.1) that have been investigated recently.

Medeiros and Menzala [8] proved the existence and uniqueness of weak solutions of the system (1.3) when $\alpha=\beta=0$ and $\gamma=\sigma=1$. These results were generalized by Medeiros and Milla Miranda [9] and Medeiros and Milla Miranda [10], that is, the authors proved existence of weak solutions for $n\geq 1$ and $\rho>-1$ for the system

$$\begin{aligned} \mathcal{A}u+|v|^{\rho+2}|u|^{\rho}u=f_1,\\ \mathcal{A}v+|u|^{\rho+2}|v|^{\rho}v=f_2. \end{aligned} \tag{1.4}$$

Besides that, the uniqueness was also established for $n=1$, 2 and 3. Many significants variations of the system (1.4) have been considered in several works, see for instance, Castro [3] and Biazutti [1].

Lately, Clark and Lima [4] established the existence and uniqueness of weak solutions for $n = 1$, 2 and 3 for the system

$$\begin{aligned} \mathcal{A}u + f(v)u &= h_1, \\ \mathcal{A}v + f(u)v &= h_2, \end{aligned} \tag{1.5}$$

where f and g are only Lipschitz functions with $f(0) = 0 = g(0)$. A another improvement of the system (1.5) was also obtained by Clark and Maciel [5]. They obtained the existence and uniqueness of weak solutions for o following

$$\mathcal{A}u_i + u_1^2 u_2^2 \cdots u_{i-1}^2 u_i u_{i+1}^2 \cdots u_n^2 = h_i \quad \text{for} \quad i = 1, 2, \cdots, n. \tag{1.6}$$

2. Existence and Uniqueness of Solutions

To establish existence and uniqueness of solutions for Cauchy problem (1.1), (1.2) we will denote by $(\cdot,\cdot)$ and $((\cdot,\cdot))$ the scalar products in $L^2(\Omega)$ and $H^1(\Omega)$ respectively. We shall use, throughout this paper, the same terminology of the functional spaces used, for instance, in the books of Lions [7].

A solution of the problem (1.1), (1.2) is understood in the following sense.

Definition 2.1. We call local weak solution of the initial-value problem (1.1), (1.2) a pair of real functions $\{u = u(x,t),\, v = v(x,t)\}$ defined for all $(x,t) \in Q_{T_0}$, where $Q_{T_0} = \Omega \times [0, T_0]$ for $T_0 > 0$ fixed, such that

$$u,\, v \text{ belong to } L^\infty\big(0, T_0; H_0^1(\Omega)\big), \tag{2.1}$$

$$u',\, v' \text{ belong to } L^2\big(0, T_0; L^2(\Omega)\big), \tag{2.2}$$

and satisfying the integral identities

$$\begin{aligned} &-\int_0^{T_0} \big(u'(t),\, \eta\big)\theta'(t)dt + \int_0^{T_0} \big(\big(u(t),\, \eta\big)\big)\theta(t)dt \\ &\qquad + \int_0^{T_0} \Big(f\Big(u(t), v(t)\Big)u(t),\, \eta\Big)\theta(t)dt = \int_0^{T_0} \big(h_1(t),\, \eta\big)\theta(t)dt \\ &\qquad\qquad\qquad \text{for all } \eta \in H_0^1(\Omega) \text{ and } \theta \in \mathcal{D}(0, T_0), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &-\int_0^{T_0} \big(v'(t),\, \eta\big)\theta'(t)dt + \int_0^{T_0} \big(\big(v(t),\, \eta\big)\big)\theta(t)dt \\ &\qquad + \int_0^{T_0} \big(g\big(u(t), v(t)\big)v(t),\, \eta\big)\theta(t)dt = \int_0^{T_0} \big(h_2(t),\, \eta\big)\theta(t)dt \end{aligned}$$

$$\text{for all } \eta \in H_0^1(\Omega) \text{ and } \theta \in \mathcal{D}(0,T_0). \quad (2.4)$$

Besides that, the initial conditions

$$u(y,0) = u_0(y) \quad \text{and} \quad v(y,0) = v_0(y), \quad (2.5)$$
$$u'(y,0) = u_1(y) \quad \text{and} \quad v'(y,0) = v_1(y), \quad (2.6)$$

are also verified.

Now we are in conditions of stating the main result of this paper, namely

Theorem 2.1. *We assume h_1 and h_2 belong to $L^2\left(0,T;L^2(\Omega)\right)$, the initial data u_0, v_0 belong to $H_0^1(\Omega)$ and u_1, v_1 belong to $L^2(\Omega)$. Besides that, if the functions f and g are $C^0(0,T;\Omega)$ in the first variable, Lipschitz in the second variable and satisfy the consistence condition*

$$f(0,0) = 0 = g(0,0),$$

then the problem (1.1), (1.2) has a unique pair of solutions $\{u, v\}$ in the sense of Definition 2.1.

To show the existence of solutions of the problem (1.1), (1.2) we will use the Galerkin's methods. Thus, by $V_n = [w_1, w_2, w_3, \cdots, w_n]$ we will denote the subspace spanned by the n first vectors of $H_0^1(\Omega)$, which is ortho-normalized and complete set in $L^2(\Omega)$. We will assume that initial data $u_n(x,0) = u_{0n}$, $v_n(x,0) = v_{0n}$, $u_n'(x,0) = u_{1n}$ and $v_n(x,0) = v_{1n}$ in V_n satisfy the following strong convergence

$$\begin{aligned} &u_{0n} \to u_0 \text{ and } v_{0n} \to v_0 \text{ in } H_0^1(\Omega),\\ &u_{1n} \to u_1 \text{ and } v_{1n} \to v_1 \text{ in } L^2(\Omega). \end{aligned} \quad (2.7)$$

Proof. Existence. The approximate problem associated with (1.1), (1.2) is given, for all $\phi \in V_n$, by the following system of ordinary differential equations

$$\begin{aligned} &\left(u_n''(t),\phi\right) + ((u_n(t),\phi)) + \left(f\left(u_n(t),v_n(t)\right),\phi\right)u_n(t) = (h_1(t),\phi),\\ &\left(v_n''(t),\phi\right) + ((v_n(t),\phi)) + \left(g\left(u_n(t),v_n(t)\right),\phi\right)v_n(t) = (h_2(t),\phi),\\ &u_n(x,0) = u_{0n} \text{ and } v_n(x,0) = v_{0n},\\ &u_n'(x,0) = u_{1n} \text{ and } v_n'(x,0) = v_{1n}, \end{aligned} \quad (2.8)$$

Under the assumption (2.7) it is known that the system (2.8) has a local solution $\{u_n, v_n\}$ on the interval $[0,t_n[$. This interval will be extended to a certain interval $[0,T_0[$, for $T_0 > 0$, thanks to the estimate below. To simplify our analysis we will write, throughout this section, u and v instead of u_n and v_n

respectively. Our next task is to determine estimates on u and v in order to take the limit in the system (2.8).

An "A Priori" Estimate. Substituting ϕ by $2u_n'(t)$ into $(2.8)_1$, ϕ by $2v_n'(t)$ into $(2.8)_2$ and adding these two equations, we have

$$\frac{d}{dt}\Big\{|u_n'(t)|^2+|v_n'(t)|^2+\|u_n(t)\|^2+\|v_n(t)\|^2\Big\}$$
$$=2\left(h_1(t),u_n'(t)\right)+2\left(h_2(t),v_n'(t)\right)$$
$$-2\big(f\big(u_n(t),v_n(t)\big)u_n(t),u_n'(t)\big)-2\big(g\big(u_n(t),v_n(t)\big)v_n(t),v_n'(t)\big),\quad (2.9)$$

where $|\cdot|$ and $||\cdot||$ denote the norm of $L^2(\Omega)$ and $H^1(\Omega)$ respectively.

Integrating (2.9) from 0 to $t\le t_n$ and observing usual inequalities we get

$$|u_n'(t)|^2+|v_n'(t)|^2+\|u_n(t)\|^2+\|v_n(t)\|^2$$
$$\le C+\int_0^t|h_1(s)|^2ds+\int_0^t|h_2(s)|^2ds+\int_0^t|u_n'(s)|^2ds+\int_0^t|v_n'(s)|^2ds$$
$$+2\int_0^t\int_\Omega\Big|f\big(u_n(s),v_n(s)\big)\Big|\,|u_n(s)|\,|u_n'(s)|dxds$$
$$+2\int_0^t\int_\Omega\Big|g\big(u_n(s),v_n(s)\big)\Big|\,|v_n(s)|\,|v_n'(s)|dxds\,,\quad (2.10)$$

where C is a real positive constant dependent only of the initial data. The two last integral of left-hand side of (2.10) can be estimated as follows: By using the Hölder inequality for $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}=1$, hypotheses on the function f, and the continuous immersion of $H_0^1(\Omega)\hookrightarrow L^q(\Omega)$ for $1\le q\le 6$ and $n=1,\,2,\,3$, we can write

$$2\int_0^t\int_\Omega\Big|f\Big(u_n(s),v_n(s)\Big)\Big|\,|u_n(s)|\,|u_n'(s)|dxds$$
$$\le C\int_0^t\int_\Omega|v_n(s)|\,|u_n(s)|\,|u_n'(s)|dxds$$
$$\le C\int_0^t|v_n(s)|_{L^6}\,|u_n(s)|_{L^3}\,|u_n'(s)|_{L^2}ds$$
$$\le C\int_0^t\|v_n(s)\|_{H_0^1}\,\|u_n(s)\|_{H_0^1}\,|u_n'(s)|_{L^2}ds$$
$$\le C\int_0^t\Big\{\|v_n(s)\|_{H_0^1}^2\,\|u_n(s)\|_{H_0^1}^2+|u_n'(s)|_{L^2}^2\Big\}ds$$
$$\le C\int_0^t\Big\{\|v_n(s)\|^4+\|u_n(s)\|^4+|u_n'(s)|^2\Big\}ds,\quad (2.11)$$

where C denotes various positive real constant independent of n and $t \leq t_n$. Analogously, we obtain

$$2\int_0^t \int_\Omega \Big| g\Big(u_n(s), v_n(s)\Big)\Big| \, |v_n(s)| \, |v_n'(s)| dx ds$$
$$\leq C \int_0^t \Big\{ \|u_n(s)\|^4 + \|v_n(s)\|^4 + |v_n'(s)|^2 \Big\} ds. \quad (2.12)$$

Taking into account (2.11) and (2.12) into (2.10) yields

$$|u_n'(t)|^2 + |v_n'(t)|^2 + \|u_n(t)\|^2 + \|v_n(t)\|^2 \leq C + C\int_0^t \Big\{ |u_n'(s)|^2 + |v_n'(s)|^2 \Big\} ds$$
$$+ C\int_0^t \Big\{ \|u_n(s)\|^4 + \|v_n(s)\|^4 \Big\} ds. \quad (2.13)$$

Denoting by $\mu(t) = \mu(u(t), v(t))$ the function defined by

$$\mu(t) = |u_n'(t)|^2 + |v_n'(t)|^2 + \|u_n(t)\|^2 + \|v_n(t)\|^2 \,,$$

we get from (2.13) that

$$\mu(t) \leq C + C\int_0^t \{\mu(s) + \mu^2(s)\} ds. \quad (2.14)$$

Now, we will use the following lemma, that will be proved later

Lemma 2.1. *Let μ a positive real function. By α, β and γ we are denoting positive real constants, with $\gamma > 1$, such that*

$$\mu(t) \leq \alpha + \beta \int_0^t \{\mu(s) + \mu^\gamma(s)\} ds.$$

Then there exists $T_0 \in \mathbb{R}$ with $0 < T_0 < T$ such that μ is bounded in $[0, T_0[$.

Thus, from (2.14) and Lemma 2.1 we can conclude

$$|u_n'(t)|^2 + |v_n'(t)|^2 + \|u_n(t)\|^2 + \|v_n(t)\|^2 \leq C,$$

for all $0 \leq t \leq T_0$ and for all n. Hence we obtain

$$\begin{array}{lll} (u_n) \ \text{ and } \ (v_n) & \text{are bounded in} & L^\infty\left(0, T_0; H_0^1(\Omega)\right), \\ (u_n') \ \text{ and } \ (v_n') & \text{are bounded in} & L^\infty\left(0, T_0; L^2(\Omega)\right). \end{array} \quad (2.15)$$

Limit of the Approximate Solutions. From estimates obtained in (2.15) we have, in particular, that the sequences (u_n), (u_n'), (v_n) and (v_n') are bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$ and $L^2\left(0,T;L^2(\Omega)\right)$, respectively. Thus, by compact injection of $H_0^1(\Omega\times]0,T[)$ into $L^2(\Omega\times]0,T[)$ it follows by Lions-Aubin's Theorem that there exist subsequences of (u_n) and (v_n), which we denote as the original sequence, such that

$$\begin{array}{llll} u_n \longrightarrow u & \text{strong in} & L^2\left(0,T;L^2(\Omega)\right) & \text{as} \quad n\longrightarrow\infty, \\ v_n \longrightarrow v & \text{strong in} & L^2\left(0,T;L^2(\Omega)\right) & \text{as} \quad n\longrightarrow\infty. \end{array}$$

Hence and as f and g are Lipschitz functions, we obtain

$$\begin{array}{llll} f\big(u_n,v_n\big)u_n \longrightarrow f\big(u,v\big)u & \text{a. e. in} \quad Q & \text{as} \quad n\longrightarrow\infty, \\ g\big(u_n,v_n\big)v_n \longrightarrow g\big(u,v\big)v & \text{a. e. in} \quad Q & \text{as} \quad n\longrightarrow\infty. \end{array} \tag{2.16}$$

On the other hand, by Hölder inequality, with $\frac{1}{p}+\frac{1}{p'}=1$ and $1\leq q<\infty$, we obtain

$$\Big|\int_Q \Big[f(u_n,v_n)u_n\Big]^q dxdt\Big| \leq \int_Q |f(u_n,v_n)|^q|u_n|^q dxdt \leq$$
$$C_f^q\int_Q |v_n|^q|u_n|^q dxdt \leq C_f^q\Big(\int_Q |u_n|^{q\cdot p}dxdt\Big)^{1/p}\Big(\int_Q |v_n|^{q\cdot p'}dxdt\Big)^{1/p'},$$

where C_f is the Lipschitz-constant associated with f. From this inequalities, if $q\cdot p\leq 6$ and $q\cdot p'\leq 6$, then $2q\leq 6$ or $1<q\leq 2$. Thus, we can affirm that

$$\big(f\left(u_n,v_n\right)u_n\big) \quad \text{is bounded in} \quad L^q\left(Q\right) \quad \text{for} \quad 1<q\leq 2. \tag{2.17}$$

By using $(2.16)_1$ and (2.17) yields

$$f\big(u_n,v_n\big)u_n \rightharpoonup f\big(u,v\big)u \quad \text{weakly in} \quad L^q(Q)$$
$$\text{as} \quad n\longrightarrow\infty \quad \text{for} \quad 1<q\leq 2. \tag{2.18}$$

Mutatis mutandis, we have

$$g\big(u_n,v_n\big)v_n \rightharpoonup g\big(u,v\big)v \quad \text{weakly in} \quad L^q(Q)$$
$$\text{as} \quad n\longrightarrow\infty \quad \text{for} \quad 1<q\leq 2. \tag{2.19}$$

We also obtain from $(2.15)_2$ the convergence

$$\begin{array}{lll} u_n \longrightarrow u & \text{weak star in} & L^\infty\left(0,T_0;H_0^1(\Omega)\right), \\ u_n' \longrightarrow u' & \text{weak star in} & L^\infty\left(0,T_0;L^2(\Omega)\right), \\ v_n \longrightarrow v & \text{weak star in} & L^\infty\left(0,T_0;H_0^1(\Omega)\right), \\ v_n' \longrightarrow v' & \text{weak star in} & L^\infty\left(0,T_0;L^2(\Omega)\right). \end{array} \tag{2.20}$$

Taking into account (2.18)-(2.20) into (2.8), and assuming that the initial data (1.2) are verified of standard way, we obtain a pair of weak solution $\{u,\, v\}$ defined on $\Omega \times [0, T_0[$ with value in $\mathbb{R}$ satisfying (1.1), (1.2) in the sense of Definition 2.1.

Uniqueness. Note that the duality $\langle u'', u' \rangle_{H^{-1}(\Omega)\times L^2(\Omega)}$ does not make sense. Thus, the standard energy method cannot be used. Therefore, the uniqueness will be gotten by Ladyzhenskaya-Visik's method, see for instance, Ladyzhenskaya and Visik [6]. Thus, let $\{u,\, v\}$ and $\{\widetilde{u},\, \widetilde{v}\}$ be two pairs of solutions of the problem (1.1), (1.2). Hence and denoting by $U = u - \widetilde{u}$ and $V = v - \widetilde{v}$ we have that the pair of functions $\{U,\, V\}$ satisfies

$$\begin{aligned} &U'' - \Delta U + f(u,v)u - f(\widetilde{u},\widetilde{v})\widetilde{u} = 0 \quad \text{in} \quad Q, \\ &V'' - \Delta V + g(u,v)v - g(\widetilde{u},\widetilde{v})\widetilde{v} = 0 \quad \text{in} \quad Q, \end{aligned} \tag{2.21}$$

$$U(0) = V(0) = 0 \quad \text{and} \quad U'(0) = V'(0) = 0. \tag{2.22}$$

Let $\phi(t)$ and $\psi(t)$ be real functions defined for all t in $]0, T_0[$ by

$$\phi(t) = \begin{cases} -\displaystyle\int_t^s U(r)dr & \text{if } \ t \le s, \\ 0 & \text{if } \ t > s, \end{cases}$$

and

$$\psi(t) = \begin{cases} -\displaystyle\int_t^s V(r)dr & \text{if } \ t \le s, \\ 0 & \text{if } \ t > s, \end{cases}$$

where $s \in]0, T_0[$. Besides that, let us consider the functions

$$\phi_1(t) = \int_0^t U(r)dr \quad \text{and} \quad \psi_1(t) = \int_0^t V(r)dr.$$

Hence and definition of the $\phi(t)$ and $\psi(t)$ yields $\phi(t) = \phi_1(t) - \phi_1(s)$ and $\psi(t) = \psi_1(t) - \psi_1(s)$ for $0 \le t \le s$.

Taking the scalar product on $L^2(\Omega)$ of ϕ with both sides of the equation $(2.21)_1$ and of ψ with both sides of the equation $(2.21)_2$ yields

$$\begin{aligned} &\int_0^s \big(U''(t),\, \phi(t)\big)\, dt + \int_0^s \big(\big(U(t),\, \phi(t)\big)\big) dt \\ &\qquad + \int_0^s \Big(f\big(u(t), v(t)\big)u(t) - f\big(\widetilde{u}(t), \widetilde{v}(t)\big)\widetilde{u}(t),\, \phi(t)\Big) dt = 0, \end{aligned} \tag{2.23}$$

$$\int_0^s \big(V''(t),\,\psi(t)\big)\,dt+\int_0^s \big(\big(V(t),\,\psi(t)\big)\big)dt$$
$$+\int_0^s \Big(g\big(u(t),v(t)\big)v(t)-g\big(\widetilde{u}(t),\widetilde{v}(t)\big)\widetilde{v}(t),\,\psi(t)\Big)dt=0. \quad (2.24)$$

Observing that

$$\int_0^s \big(U''(t),\,\phi(t)\big)\,dt=-\frac{1}{2}|U(s)|^2$$
$$\text{and}\quad \int_0^s \big(\big(U(t),\,\phi(t)\big)\big)dt=-\frac{1}{2}\|\phi_1(s)\|^2,$$

then the equation (2.23) can be written as

$$\frac{1}{2}|U(s)|^2+\frac{1}{2}\|\phi_1(s)\|^2$$
$$=\int_0^s \Big(f\big(u(t),v(t)\big)u(t)-f\big(\widetilde{u}(t),\widetilde{v}(t)\big)\widetilde{u}(t),\,\phi(t)\Big)dt. \quad (2.25)$$

Adding and subtracting the term $f(u,v)\widetilde{u}$ in the integral of right-hand side of (2.25) and observing that f is a Lipschitz function we obtain

$$\int_0^s \Big(f\big(u(t),v(t)\big)u(t)-f\big(\widetilde{u}(t),\widetilde{v}(t)\big)\widetilde{u}(t),\,\phi(t)\Big)dt\le$$
$$\int_0^s \Big|\Big(f\big(u(t),v(t)\big)U(t),\,\phi(t)\Big)\Big|dt+\int_0^s \Big(C_f|V(t)|\widetilde{u}(t),\,\phi(t)\Big)dt. \quad (2.26)$$

By using the Hölder inequality and Sobolev's Embedding Theorem for $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1,$ yields

$$\Big(f\big(u(t),v(t)\big)U(t),\,\phi(t)\Big)\le\int_\Omega |f\big(u(t),v(t)\big)|\,|U(t)|\,|\phi(t)|dx$$
$$\le\Big(\int_\Omega |U(t)|^2dx\Big)^{1/2}\Big(\int_\Omega C_f^3|v(t)|^3dx\Big)^{1/3}\Big(\int_\Omega |\phi(t)|^6dx\Big)^{1/6}$$
$$\le C_1|U(t)|_{L^2(\Omega)}\|\phi(t)\|_{H_0^1(\Omega)}. \quad (2.27)$$

Analogously, we also find

$$\Big(C_f|V(t)|\widetilde{u}(t),\,\phi(t)\Big)\le C_2|V(t)|_{L^2(\Omega)}\|\phi(t)\|_{H_0^1(\Omega)}. \quad (2.28)$$

Taking into account (2.27) and (2.28) into (2.25) yields

$$\frac{1}{2}|U(s)|^2 + \frac{1}{2}\|\phi_1(s)\|^2 \\ \leq C\int_0^s \Big(|U(t)|_{L^2(\Omega)} + |V(t)|_{L^2(\Omega)}\Big)\|\phi(t)\|_{H_0^1(\Omega)}\,dt. \quad (2.29)$$

Similarly, we also get

$$\frac{1}{2}|V(s)|^2 + \frac{1}{2}\|\psi_1(s)\|^2 \\ \leq C\int_0^s \Big(|U(t)|_{L^2(\Omega)} + |V(t)|_{L^2(\Omega)}\Big)\|\psi(t)\|_{H_0^1(\Omega)}\,dt. \quad (2.30)$$

As $\phi(t) = \phi_1(t) - \phi_1(s)$ then we obtain from (2.29) that

$$\frac{1}{2}|U(s)|^2 + \frac{1}{2}\|\phi_1(s)\|^2 \\ \leq C\int_0^s \Big[\Big(|U(t)|_{L^2(\Omega)} + |V(t)|_{L^2(\Omega)}\Big)^2 + \|\phi_1(t)\|^2_{H_0^1(\Omega)}\Big]dt \\ + \frac{C}{2}\int_0^s \Big(|U(t)|_{L^2(\Omega)} + |V(t)|_{L^2(\Omega)}\Big)^2 dt + \frac{C_s}{2}\|\phi_1(s)\|^2_{H_0^1(\Omega)}.$$

or

$$\frac{1}{2}|U(s)|^2 + \Big(\frac{1}{2} - \frac{C_s}{2}\Big)\|\phi_1(s)\|^2 \\ \leq C\int_0^s \Big(|U(t)|^2_{L^2(\Omega)} + |V(t)|^2_{L^2(\Omega)} + \|\phi_1(t)\|^2_{H_0^1(\Omega)}\Big)dt, \quad (2.31)$$

where C represents various positive real constants. Analogously, we obtain from (2.24) that

$$\frac{1}{2}|V(s)|^2 + \Big(\frac{1}{2} - \frac{C_s}{2}\Big)\|\psi_1(s)\|^2 \\ \leq C\int_0^s \Big(|U(t)|^2_{L^2(\Omega)} + |V(t)|^2_{L^2(\Omega)} + \|\psi_1(t)\|^2_{H_0^1(\Omega)}\Big)dt. \quad (2.32)$$

Adding (2.31) with (2.32) we can write

$$\frac{1}{2}\Big[|U(s)|^2 + |V(s)|^2 + \Big(\frac{1}{2} - \frac{C_s}{2}\Big)\Big(\|\phi_1(s)\|^2 + \|\psi_1(s)\|^2\Big)\Big] \\ \leq C\int_0^s \Big(|U(t)|^2_{L^2(\Omega)} + |V(t)|^2_{L^2(\Omega)} + \|\phi_1(t)\|^2_{H_0^1(\Omega)} + \|\psi_1(t)\|^2_{H_0^1(\Omega)}\Big)dt. \quad (2.33)$$

Choosing s_0 in $[0,T_0[$ such that $\frac{1}{2}+\frac{C_{s_0}}{2}=\frac{1}{4}$ and using the Gronwall's inequality in (2.33) we obtain

$$U(t)=V(t)=0 \quad \text{for all} \quad 0\le t\le s_0.$$

Observe that for $t\in[0,s_0]$, we have

$$\frac{1}{2}-\frac{tC}{2}\ge\frac{1}{2}-\frac{s_0C}{2}=\frac{1}{4}.$$

If $s_0=T_0$ we will stop. But, if $s_0<T_0$ we will use the same argument with the initial data zero in the place of s_0, i.e.,

$$U(s_0)=V(s_0)=0 \qquad \text{and} \qquad U'(s_0)=V'(s_0)=0.$$

In this conditions, we consider $s_0<s<T_0$ and proceed as in the previous case to obtain $|U(t)|=|V(t)|=0$ in $[s_0,2s_0]$. Thus, proceeding this way, we can conclude that $U(t)=V(t)=0$ a.e. in $[0,T_0[$. What it implicates $u=\widetilde{u}$ and $v=\widetilde{v}$ a.e. in $[0,T_0[$. Therefore, we have that the proof of Theorem 2.1 is ended. □

Finally, as said before, we will do the proof of Lemma 2.1.

Proof of Lemma 2.1. Let θ be the function defined by $\theta(t)=\displaystyle\int_0^t\big(\mu(s)+\mu^\gamma(s)\big)ds$. Hence it follows that $\theta(0)=0$ and

$$\theta'(t)=\mu(t)+\mu^\gamma(t)\le\alpha+\beta\theta(t)+\big(\alpha+\beta\theta(t)\big)^\gamma. \tag{2.34}$$

Multiplying (2.34) by $\exp(-\beta t)$ and integrating from 0 to $t<T$ yields

$$\begin{aligned}\theta(t)=\exp(\beta t)\int_0^t\Big[\alpha+\Big(\alpha+\beta\theta(s)\Big)^\gamma\Big]ds&\le\alpha T\exp(\beta T)\\&+\exp(\beta T)\int_0^t\Big(\alpha+\beta\theta(s)\Big)^\gamma ds=a+b\int_0^t\Big(\alpha+\beta\theta(s)\Big)^\gamma ds\,,\end{aligned}$$

where $a=\alpha T\exp(\beta T)$ and $b=\exp(\beta T)$. Denoting by $\xi(t)=\displaystyle\int_0^t\big(\alpha+\beta\theta(s)\big)^\gamma ds$ we have $\xi(0)=0$ and

$$\xi'(t)=\Big(\alpha+\beta\theta(t)\Big)^\gamma\le\Big[\alpha+\beta\Big(a+b\xi(t)\Big)\Big]^\gamma.$$

Dividing this inequalities by the term of the right-hand side we get

$$\frac{\xi'(t)}{\Big[\alpha+\beta\Big(a+b\xi(t)\Big)\Big]^\gamma}\le 1.$$

Integrating this expression from 0 to t and to proceed applying the fundamental theorem of calculus in the left-hand side of the resulting expression, we obtain

$$\frac{1}{\beta b}\left\{\frac{1}{1-\gamma}\left[\alpha+\beta\Big(a+b\xi(s)\Big)\right]^{1-\gamma}\right\}\Big|_{s=0}^{s=t}\leq t\,,$$

which implicates

$$\frac{1}{1-\gamma}\left[\alpha+\beta\Big(a+b\xi(t)\Big)\right]^{1-\gamma}\leq\frac{1}{1-\gamma}\Big(\alpha+\beta a\Big)^{1-\gamma}+\beta t.$$

Hence and being $\gamma>1$ we can write

$$\left[\alpha+\beta\Big(a+b\xi(t)\Big)\right]^{1-\gamma}\geq\Big(\alpha+\beta a\Big)^{1-\gamma}-\beta b(\gamma-1)t.$$

Assuming that $\Big(\alpha+\beta a\Big)^{1-\gamma}-\beta b(\gamma-1)t>0,$ then

$$\left[\alpha+\beta\Big(a+b\xi(t)\Big)\right]^{\gamma-1}\leq\left[\Big(\alpha+\beta a\Big)^{1-\gamma}-\beta b(\gamma-1)t\right]^{-1},$$

and therefore

$$\alpha+\beta\Big(a+b\xi(t)\Big)\leq\left\{\left[\frac{(\alpha+\beta a)^{1-\gamma}}{\beta b(\gamma-1)}-t\right]^{\frac{1}{1-\gamma}}\cdot\Big[\beta b(\gamma-1)\Big]^{\frac{1}{1-\gamma}}\right\}^{-1}.$$

Choosing T_0 such that $0<T_0<\frac{(\alpha+\beta a)^{1-\gamma}}{\beta b(\gamma-1)},$ then there exists a positive real constant K_0 satisfying

$$\alpha+\beta\Big(a+b\xi(t)\Big)\leq K_0\ \text{ for all }\ t\ \text{ in }\ [0,T_0], \tag{2.35}$$

where

$$K_0=\left\{\left[\frac{(\alpha+\beta a)^{1-\gamma}}{\beta b(\gamma-1)}-T_0\right]^{\frac{1}{\gamma-1}}\cdot\Big[\beta b(\gamma-1)\Big]^{\frac{1}{1-\gamma}}\right\}^{-1}.$$

From (2.35) and observing that $\theta(t)\leq a+b\xi(t)$ it follows $\theta(t)\leq K_1$ for all t in $[0,T_0]$. Finally, being $\mu(t)\leq\alpha+\beta\theta(t),$ we have $\mu(t)\leq K_2$ for all t in $[0,T_0]$. Thus, we conclude the proof of Lemma 2.1. □

3. Some Applications and Comments

As applications of the Theorem 2.1 we have the two particular cases:

I. Let us consider m and n real functions with real value given by $m(x) = n(x) = x^2$. The function x^2 is a locally-Lipschitz function in the bounded-open sets of the $\mathbb{R}$. Thus, taking $f(u,v) = v^2$ and $g(u,v) = u^2$, we conclude as a consequence of Theorem 2.1 that the Cauchy problem associated with

$$\begin{aligned} u'' - \Delta u + v^2 u &= h_1, \\ v'' - \Delta v + u^2 v &= h_2 \end{aligned}$$

has a unique solution in the sense of Definition 2.1. For more details, see Medeiros-Menzala [8] .

II. Setting $f(u,v) = |u|^\rho |v|^{\rho+2}$, for $\rho \geq -1$, and $g(u,v) = |u|^{\rho+2}|v|^\rho$, for $\rho \geq 1$, we can conclude that both are Lipschitz functions in the bounded-open sets of $\mathbb{R}^2$. In fact,

$$|f(u,v_1) - f(u,v_2)| = |u|^\rho \Big| \, |v_1|^{\rho+2} - |v_2|^{\rho+2} \, \Big|,$$

and if $L_1(w) = |w|^{\rho+2}$ then

$$\left| L_1'(w) \right| = \left| (\rho+2)\,|w|^{\rho+1} \frac{w}{|w|} \right| = (\rho+2)\,|w|^{\rho+1} \leq K,$$

in the bounded set of $\mathbb{R}^n$ if $\rho \geq -1$. Besides that $f(0,0) = 0$. In a similar way, for $\rho \geq 0$ the function g satisfies

$$\begin{aligned} |g(u,v_1) - g(u,v_2)| = |u|^{\rho+2} \Big| \, |v_1|^\rho - |v_2|^\rho \, \Big| \\ \leq K|u|^{\rho+2}|v_1 - v_2| \leq K|v_1 - v_2|. \end{aligned}$$

On the other hand, if $L_2(w) = |w|^\rho$ then $|L_2'(w)| = \rho\,|w|^{\rho-1} \leq K$ for $\rho \geq 1$, this way we have the previously statements. Note that, if $\rho \geq \max\{-1, 1\} = 1$ then it implies f and g are locally Lipschitz functions. In this conditions, the Theorem 2.1 guarantees that the Cauchy problem for the system

$$\begin{aligned} u'' - \Delta u + |u|^\rho |v|^{\rho+2} u &= h_1, \\ v'' - \Delta v + |u|^{\rho+2}|v|^\rho v &= h_2, \end{aligned}$$

with $\rho \geq 1$ has a unique solutions in sense of Definition 2.1. This model was investigated by Medeiros and Milla Miranda [10].

Final Remark. We concluded this paper observing that the system (1.1) can be included in the following natural framework

$$\begin{gathered} u_1'' - \Delta u_1 + f_1(u_1, u_2, \cdots, u_k)u_1 = h_1, \\ u_2'' - \Delta u_2 + f_2(u_1, u_2, \cdots, u_k)u_2 = h_2, \\ \vdots \\ u_k'' - \Delta u_k + f_k(u_1, u_2, \cdots, u_k)u_k = h_k, \end{gathered}$$

and the Cauchy problem associated with this system can be investigated in a similar way as made in the problem (1.1), (1.2). □

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