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PROJECTIVE VARIETIES CONTAINING NO NON-REFLEXIVE CURVE

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Abstract: Let $X \subset \mathbf{P}^n$ be an integral variety. Here we give conditions on X which allow us to describe all non-reflexive curves $C \subset X$ (e.g. to show that X contains no non-reflexive curve).

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1. Introduction

We work over an algebraically closed field $\mathbb{K}$ such that $p := \mathrm{char}(\mathbb{K}) > 0$. Except in Remark 5 we will always assume $p \neq 2$. Let $C \subset \mathbf{P}^n$ be an integral curve which is not a line. C is non-reflexive if and only if for a general $P \in C_{reg}$ the Zariski tangent line $T_PC \subset \mathbf{P}^n$ contains the effective divisor $3P$ of C (see [6], 3.5, or [3], 2.1); if this is the case, then $T_PC \subset \mathbf{P}^n$ contains the effective divisor pP of C (see [3], 2.1) and this is true for every $P \in C_{reg}$.

Notation 1. Let $X \subset \mathbf{P}^n$ be an integral m-dimensional variety. For every integer $t \geq 1$ let tP_X denote the infinitesimal neighborhood of order $(t-1)$ of P in X, i.e. the closed subscheme of X with $(\mathcal{I}_{X,P})^t$ as its ideal sheaf. Hence tP_X is a zero-dimensional subscheme of X, $(tP_X)_{red} = \{P\}$ and $\mathrm{length}(tP_X) = \binom{m+t-1}{m}$. Set $\alpha(X, 3P) := \dim(\langle 3P_X \rangle)$. Hence $\alpha(X, 3P) \leq \min\{n, (m^2 + 3m)/2)\}$. Let $\alpha(X, 3)$ denote the supremum of all integers $\alpha(X, 3P)$ for $P \in X_{reg}$. By semicontinuity we have $\alpha(X, 3) = \alpha(X, Q)$ for a general $Q \in X_{reg}$.

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Hence $\alpha(X,3) \leq \min\{n, (m^2+3m)/2)\}$. Since X is embedded in $\mathbf{P}^n$ we also have $\alpha(X,3) \geq m$.

In Section 2 we will prove the following result and give a few related results.

Theorem 1. *Let $X \subset \mathbf{P}^n$ be an integral non-degenerate m-dimensional variety. Assume $n \geq \max\{2m+2, m+1+\alpha(X,3)\}$. Fix an integer x such that $1 \leq x \leq n-m-\max\{m+1, \alpha(X)\}$. Let $V \subset \mathbf{P}^n$ be a general $(x-1)$ dimensional linear subspace. Let $f : \mathbf{P}^n \backslash V \to \mathbf{P}^{n-x}$ be the linear projection from V. Then $f|X$ is injective, $f(\mathrm{Sing}(X)) = \mathrm{Sing}(f(X))$, $f|X_{reg}$ has differential of rank m. An integral curve $C \subseteq f(X)$ such that $C \nsubseteq \mathrm{Sing}(f(X))$ is non-reflexive if and only if there is an integral non-reflexive curve $D \subseteq X$ such that $f(D) = C$. In this case D is unique, $D = f^{-1}(C) \cap X$ and $D \nsubseteq \mathrm{Sing}(X)$.*

As an immediate corollary of Theorem 1 and the inequality $\alpha(X,3) \leq (m^2+3m)/2$ we obtain the following result.

Corollary 1. *Let $X \subset \mathbf{P}^n$ be an integral non-degenerate m-dimensional variety. Assume $n \geq \max\{m+1+(m^2+3m)/2\}$. Fix an integer x such that $1 \leq x \leq n-m-(m^2+3m)/2$. Let $V \subset \mathbf{P}^n$ be a general $(x-1)$ dimensional linear subspace. Let $f : \mathbf{P}^n \backslash V \to \mathbf{P}^{n-x}$ be the linear projection from V. Then $f|X$ is injective, $f(\mathrm{Sing}(X)) = \mathrm{Sing}(f(X))$, $f|X_{reg}$ has differential of rank m. An integral curve $C \subseteq f(X)$ such that $C \nsubseteq \mathrm{Sing}(f(X))$ is non-reflexive if and only if there is an integral non-reflexive curve $D \subseteq X$ such that $f(D) = C$. In this case D is unique, $D = f^{-1}(C) \cap X$ and $D \nsubseteq \mathrm{Sing}(X)$.*

We now list explicitly the particular cases of Theorem 1 and Corollary 1 in whixh we assume that $\mathrm{Sing}(X)$ is finite (or empty).

Proposition 1. *Let $X \subset \mathbf{P}^n$ be an integral non-degenerate m-dimensional variety with at most isolated singularities. Assume $n \geq \max\{2m+2, m+1+\alpha(X,3)\}$. Fix an integer x such that $1 \leq x \leq n-m-\max\{m+1, \alpha(X)\}$. Let $V \subset \mathbf{P}^n$ be a general $(x-1)$ dimensional linear subspace. Let $f : \mathbf{P}^n \backslash V \to \mathbf{P}^{n-x}$ be the linear projection from V. Then $f|X$ is injective, $f(\mathrm{Sing}(X)) = \mathrm{Sing}(f(X))$ and $f|X_{reg}$ has differential of rank m. An integral curve $C \subseteq f(X)$ such that $C \nsubseteq \mathrm{Sing}(f(X))$ is non-reflexive if and only if there is an integral non-reflexive curve $D \subseteq X$ such that $f(D) = C$. In this case D is unique and $D = f^{-1}(C) \cap X$.*

Proposition 2. *Let $X \subset \mathbf{P}^n$ be an integral non-degenerate m-dimensional variety with at most isolated singularities. Assume $n \geq \max\{m+1+(m^2+3m)/2$. Fix an integer x such that $1 \leq x \leq n-m-(m^2+3m)/2\}$. Let $V \subset \mathbf{P}^n$ be a general $(x-1)$ dimensional linear subspace. Let $f : \mathbf{P}^n \backslash V \to \mathbf{P}^{n-x}$ be* *the linear projection from V. Then $f|X$ is injective, $f(\mathrm{Sing}(X)) = \mathrm{Sing}(f(X))$, $f|X_{reg}$ has differential of rank m. An integral curve $C \subseteq f(X)$is non-reflexive if and only if there is an integral non-reflexive curve $D \subseteq X$ such that $f(D) = C$. In this case D is unique and $D = f^{-1}(C) \cap X$.*

Remark 1. Fix an integer $z \geq 2$. Let $Y \subset \mathbf{P}^r$ be an integral m-dimensional variety. Let $X \subset \mathbf{P}^n$, $n := \binom{r+t}{r} - 1 - h^0(\mathbf{P}^r, \mathcal{I}_Y(t))$, be the non-degeneate embedding obtained composing the given embedding of Y in $\mathbf{P}^r$ with the order t Veronese embedding of $\mathbf{P}^r$. X has no non-reflexive curve (see [4], Theorem 2.5, or [5], Theorem 20). Hence we may apply Theorem 1, Corollary 1 and Propositions 1, 2 to obtain several non-complete embeddings of every integral variety having no non-reflexive curve.

2. Proof of Theorem 1

Lemma 1. *Let $X \subset \mathbf{P}^n$ be an integral variety and $P \in X_{reg}$. Assume $\dim(\langle Z \rangle) = 2$ for all length 3 zero-dimensional schemes $Z \subset X$ such that $Z_{red} = \{P\}$. Then there is no integral non-reflexive curve $C \subset X$ such that $P \in C$.*

Proof. First, we will show that there is no non-reflexive curve $C \in X$ such that $P \in C_{reg}$. The Zariski tangent line $T_PC \subset \mathbf{P}^n$ of C at P has order of contact at least p (and hence at least 3) with C at P (see [6], 3.5, or [3], 2.1) and hence there is a length 3 zero-dimensional scheme $Z \subset C \subset X$ such that $\langle Z \rangle = T_PC$ is a line, contradiction. Now assume $P \in \mathrm{Sing}(C)$. Let $G(1,n)$ denote the Grassmannian of all lines contained in $\mathbf{P}^n$. For every $Q \in C_{reg}$ the tangent line T_QC to C at Q has order of contanct at least $p \geq 3$ with C at Q. Since $G(1,n)$ is complete. the family $\{T_QC\}_{Q \in X_{reg}}$ has at least one limit $D \in C$ such that $P \in C$. By semicontinuity the scheme $D \cap C$ has a connected component Z such that $Z_{red} = \{P\}$ and $\mathrm{length}(Z) \geq p \geq 3$, contradiction. □

As an immediate corollary of Lemma 1 we obtain the following result.

Corollary 2. *Let X be a smooth projective variety and $L \in \mathrm{Pic}(X)$. If L is 2-spanned, then the image of X by the complete linear system $|L|$ contains no non-reflexive curve.*

Notice that a 2-very ample line bundle in the sense of [2] or [7] is 2-spanned in the sense of [1].

Corollary 3. *Let $X \subset \mathbf{P}^n$ be an integral surface. Fix a general $P \in X$. Assume $\dim(\langle Z \rangle) = 2$ for a general $P \in X$ every length 3 zero-dimensional scheme such that $Z_{red} = \{P\}$. Then X contains at most finitely many non-reflexive curves.*

The proof of Lemma 1 gives verbatim the following result.

Proposition 3. *Let $X \subset \mathbf{P}^n$ be an integral surface with only isolated singularities. Assume $\dim(\langle Z \rangle) = 2$ for every connected length 3 zero-dimensional scheme $Z \subset X_{reg}$. Then X contains no non-reflexive curve.*

Remark 2. Let $X \subset \mathbf{P}^n$ an integral variety. Assume the existence of a non-reflexive integral curve $C \subseteq X$, C not a line. Assume $\deg(X) < p$. Then for every $P \in C_{reg}$ the variety X contains the line $T_P C$ (Bezout). Hence X contains infinitely many lines.

Remark 3. Let $X \subset \mathbf{P}^n$ an integral variety. Assume the existence of a non-reflexive integral curve $C \subseteq X$, C not a line. Assume that X is set-theoretically cut out by hypersurfaces of degree at most $p-1$. Then for every $P \in C_{reg}$ the variety X contains the line $T_P C$ (Bezout). Hence X contains infinitely many lines.

Now we list without proof a few results which are easy consequences of the previous statements and of published results. Proposition 4 follows from Corollary 2 and [7], part (ii) of Corollary at p. 176 for $d = 2$.

Proposition 4. *Let X be a minimal surface of general type and L an ample line bundle on X. Then $\phi_{t(\omega_X+L)}$ is very ample and $\phi_{t(\omega_X+L)}(X)$ contains no non-reflexive curve for every integer $t \geq 3$.*

Remark 4. Let X be a smooth Fano threefold such that $b_2(X) = 1$. By [8] the classification is the same as in the characteristic zero case. In particular the classical classification of all Fano threefolds with index ≥ 2 is the same and show that for every $P \in X$ there are non-reflexive curves $C \subset X$ such that $P \in C_{reg}$ and that for every line $D \subset X$ such that $P \in D$ there is a reflexive curve $C \subset X$ such that $P \in C_{reg}$ and D is tangent to C at P. By Remark 3 and the assumption $p \geq 3$ there are no other non-reflexive curves contained in X. By [8], Corollary 4.4, if $g \geq 1$ and X has index one, then $|-K_X|$ is very ample and the anti-canonical model of X is cut out by quadrics, even the structure of the lines in it are quite understood (see [8]). For $g \geq 4$ one should use $|-2K_X|$ (and $p \geq 5$) or $|-3K_X|$ (again, $p \geq 5$ is enough).

Proof of Theorem 1. Let $\mathrm{Sec}(X) \subseteq \mathbf{P}^n$ be the secant variety of X and $TX \subseteq \mathbf{P}^n$ be the closure in $\mathbf{P}^n$ of the union of all Zariski tangent spaces T_PX for $P \in X_{reg}$. Hence $\mathrm{Sec}(X)$ and TX are irreducible, $\dim(\mathrm{Sec}(X)) \leq 2m+1$ and $\dim(TX) \leq 2m$. By the generality of V we have $V \cap \mathrm{Sec}(X) = \emptyset$ and $V \cap TX = \emptyset$. Hence $f|X$ is injective, $f(\mathrm{Sing}(X)) = \mathrm{Sing}(f(X))$ and $f|X_{reg}$ has differential of rank m. Let $T^2(X)$ denote the closure in $\mathbf{P}^n$ of all $\langle 3P_X \rangle$ for all $P \in X_{reg}$. Hence $T^2(X)$ is irreducible and $\dim(T^2(X)) \leq \alpha(X,3)+m$. By the generality of V and the inequality $x \leq n-m-\max\{m+1, \alpha(X)\}$ we have $V \cap T^2(X) = \emptyset$. Hence the theorem follows from Lemma 1. □

Remark 5. Assume $p = 2$. Let $C \subset \mathbf{P}^n$ be an integral projective curve. C is non-reflexive, but it is natural to say that C is non-ordinary if the tangent line of C at a general point of C has an order of contact at least 3 with C at P. In this case (as in the case $p \geq 3$) the order of contact is a power q of the characteristic. Hence in this case we have $q \geq 4$. We may extend verbatim to this situation speaking of non-ordinary curves instead of non-reflexive curves. To use Remark 1 for $p = 2$, use [6], 5.4, i.e. [5], Theorem 20 (ii).

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