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ON THE MORDUKHOVICH SUBDIFFERENTIAL IN BINORMED SPACES AND SOME APPLICATIONS

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Abstract: The limiting subdifferential was first studied by Mordukhovich in [5], followed by joint work with Kruger in [4], and by work of Ioffe [2, 3].

The power of the limiting subdifferential as a tool in recognizing metric regularity was first observed by Mordukhovich [6]. Using the limiting subdifferential, he presented a convenient test for the metric regularity of strictly differentiable mappings in terms of the adjoint of its strict derivative.

In this paper, we give a test for the metric regularity of non necessarily strictly differentiable mappings using the notion of Mordukhovich subdifferentiability in binormed spaces. This result strengthens and generalizes the elegant result of Mordukhovich.

Finally, we give an example of non strictly differentiable mapping for which the given test works.

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1. An Introduction to Metric Regularity and Mordukhovich Subdifferential

Throughout this section we consider an open set U of a separable Banach space $(E, \| \, . \, \|_1)$, a closed set $S \subset U$, a Euclidean space $(Y, \| \, . \, \|)$, a $\| \, . \, \|_1$-locally Lipschitz map $h : U \longrightarrow Y$ around the point $x \in U$, and a $\| \, . \, \|_1$-locally Lipschitz real function h_1 around the point x.

Let d_S^1 be the distance function to the set S with respect to the norm $\| \, . \, \|_1$, and $\partial_-^1 h_1(x)$ be the $\| \, . \, \|_1$-Dini Subdifferential of h_1 at x.

The Mordukhovich (Limiting) subdifferential of h_1 at x with respect to the norm $\| \, . \, \|_1$, denoted $\partial_a^1 h_1(x)$, is given by

$$\partial_a^1 h_1(x) = \{\lim \phi_r : \ x_r \rightarrow_{\|.\|_1} x, \ \phi_r \in \partial_-^1 h_1(x_r)\},$$

where lim denotes $*\sigma((E, \| \, . \, \|_1)', E)$ limit.

Analogously, the Mordukhovich normal cone to a set S at a point $x \in S$ with respect to the norm $\| \, . \, \|_1$, denoted $N_S^{a,1}(x)$, is given by $N_S^{a,1}(x) = R^+ \partial_a^1 d_S^1(x)$.

Let $T_S^1(x)$ be the Clarke tangent cone to S at x with respect to the norm $\| \, . \, \|_1$.

Let h_2 a $\| \, . \, \|_2$-locally Lipschitz real function around the point x. The generalized Clarke subdifferential of h_2 with respect to the pair of norms $(\| \, . \, \|_1 , \| \, . \, \|_2)$, denoted $\partial_0^{1,2} h_2(x)$, is the subset of $(E, \| \, . \, \|_2)'$ given by

$$\partial_0^{1,2} h_2(x) = \{\xi \in (E, \| \, . \, \|_2)' : \ h_2^{0,1}(x, v) \geq < \xi, v > \ \ \forall v \in E\},$$

where $h_2^{0,1}(x, v)$ is the Clarke directional derivative of h_2 at x in the direction v with respect to the norm $\| \, . \, \|_1$.

Notice that the Clarke subdifferential $\partial_0^2 h_2(x)$ is smaller than the generalized Clarke subdifferential $\partial_0^{1,2} h_2(x)$.

For a point x in S, the generalized Clarke tangent cone to S at x with respect to the pair of norms $(\| \, . \, \|_1, \| \, . \, \|_2)$, denoted $T_S^{1,2}(x)$, is the subset of E given by $T_S^{1,2}(x) = (\mathrm{cl}(R^+ \partial_0^{1,2} d_S^2(x)))^-$, where cl denotes $*\sigma((E, \| \, . \, \|_2)', E)$ closure.

Analogously, for a point x in S, the generalized Clarke normal cone to S at x with respect to the pair of norms $(\| \, . \, \|_1, \| \, . \, \|_2)$, denoted $N_S^{1,2}(x)$, is the subset of $(E, \| \, . \, \|_2)'$ given by $N_S^{1,2}(x) = cl(R^+ \partial_0^{1,2} d_S^2(x))$, where cl denotes $*\sigma((E, \| \, . \, \|_2)', E)$ closure.

Assume now that $x \in S$. We say h is $\| \, . \, \|_1$-metrically regular on S at x if there is a real constant $K > 0$ such that

$$d^1_{S\cap h^{-1}(y)}(z) \leq k \parallel h(z) - y \parallel \text{ for all } z \text{ close to } x$$
$$\text{with respect to the norm } \parallel . \parallel_1$$

and all vector $y \in Y$ close to $h(x)$.

If h is $\parallel . \parallel_1$-strictly differentiable at x, we can present a convenient test for the metric regularity of h in terms of Mordukhovich normal cone.

Theorem 1. *If h is $\parallel . \parallel_1$-strictly differentiable at $x \in S$ and $((\nabla h(x))^*)^{-1}(N_S^{a,1}(x)) = \{0\}$ or, in particular, $\nabla h(x)T_S^1(x) = Y$ then h is $\parallel . \parallel_1$-metrically regular on S at x.*

Using the notion of strict differentiability with respect to pair of norms $(\parallel . \parallel_1, \parallel . \parallel_2)$ and the notion of Mordukhovich subdifferential in Binormed spaces, we want to establish the same conclusion of Theorem 1 for non necessarily $\parallel . \parallel_1$-strictly differentiable mappings.

For this, we pause to recall some terminology before proving the main result.

2. Strictly Differentiable Mappings with Respect to the Pair of Norms $(\parallel . \parallel_1, \parallel . \parallel_2)$

Let $\parallel . \parallel_1$, $\parallel . \parallel_2$ two norms defined on a linear space E such that $(E, \parallel . \parallel_2)$ is a separable Banach space and for some $c > 0$ $\parallel . \parallel_1 \leq c \parallel . \parallel_2$.

Let U an open subset of $(E, \parallel . \parallel_1)$, S a closed subset of $(E, \parallel . \parallel_1)$ such that $S \subset U$, $(Y, \parallel . \parallel)$ a Euclidean space, $h : U \longrightarrow Y$.

We say h is $(\parallel . \parallel_1, \parallel . \parallel_2)$-strictly differentiable at $x \in U$ if there exists a continuous linear operator L from $(E, \parallel . \parallel_2)$ into $(Y, \parallel . \parallel)$ such that

$$h(x' + s) = h(x') + Ls + o(\parallel s \parallel_2) \text{ as } x' \to_{\parallel . \parallel_1} x, \quad s \to_{\parallel . \parallel_1} 0.$$

Let us remark that such a L is unique since h is $\parallel . \parallel_2$- Fréchet differentiable at x.

If h is $(\parallel . \parallel_1, \parallel . \parallel_2)$-strictly differentiable at $x \in U$, then we set $\nabla h(x) = L$.

3. Mordukhovich Subdifferential in Binormed Spaces and Regularity

Throughout this section we consider a binormed space $(E, \parallel . \parallel_1, \parallel . \parallel_2)$ such that $(E, \parallel . \parallel_2)$ is a separable Banach space and for some $c > 0$ $\parallel . \parallel_1 \leq c \parallel . \parallel_2$, an open set U of normed space $(E, \| \,.\, \|_1)$, a closed subset S of $(E, \| \,.\, \|_1)$ such that $S \subset U$, a Euclidean space $(Y, \| \,.\, \|)$, a locally Lipschitz map $h : U \longrightarrow (Y, \| \,.\, \|)$ with respect to the norm $\| \,.\, \|_2$, and a $\| \,.\, \|_2$-locally Lipschitz real function h_2 around $\bar{x} \in U$.

We define the generalized Mordukhovich subdifferential of h_2 at $\bar{x}$ with respect to the pair of norms $(\| \,.\, \|_1, \| \,.\, \|_2)$, denoted $\partial_a^{1,2} h_2(\bar{x})$ by:

$$\partial_a^{1,2} h_2(\bar{x}) = \{\lim \phi_r : \ x_r \rightarrow_{\|.\|_1} \bar{x}, \ \phi_r \in \partial_-^2 h_2(x_r)\},$$

where lim denotes $*\sigma((E, \| \,.\, \|_2)', E)$ limit.

By $\| \,.\, \|_1$-upper semicontinuity of the Clarke directional derivative $h_2^{0,1}$, we deduce that if h_2 is $(\| \,.\, \|_1, \| \,.\, \|_2)$-strictly differentiable at x then $\partial_a^{1,2} h_2(x) = \{\nabla h_2(x)\}$.

Let us remark that the Mordukhovich subdifferential $\partial_a^1 h_2(\bar{x})$ is smaller than the generalized Mordukhovich subdifferential $\partial_a^{1,2} h_2(\bar{x})$.

Analogously, we define the Mordukhovich normal cone to a set S at a point $x \in S$ with respect to the pair of norms $(\| \,.\, \|_1, \| \,.\, \|_2)$, denoted $N_S^{a,1,2}(x)$ to be the subset of $(E, \| \,.\, \|_2)'$ given by $N_S^{a,1,2}(x) = R^+ \partial_a^{1,2} d_S^2(x)$.

The first assertion below reiterates that if h_2 is a $\| \,.\, \|_2$-locally Lipschitz map from U to R, then the pair of multifunctions $(\partial_a^2 h_2, \partial_a^{1,2} h_2)$ is closed from $(U, \| \,.\, \|_1)$ to $((E, \| \,.\, \|_2)', *\sigma((E, \| \,.\, \|_2)', E))$ in the following sens: if x_i and ξ_i are sequences in U and $(E, \| \,.\, \|_2)'$ such that $\xi_i \in \partial_a^2 h_2(x_i)$, x_i converges to x with respect to the norm $\| \,.\, \|_1$, and ξ_i converges to ξ for the weak topology $*\sigma((E, \| \,.\, \|_2)', E)$, then $\xi \in \partial_a^{1,2} h_2(x)$.

Lemma 2. *Assume that h_2 is a $\| \,.\, \|_2$-locally Lipschitz map from U to R. Then the pair of multifunctions $(\partial_a^2 h_2, \partial_a^{1,2} h_2)$ is closed from $(U, \| \,.\, \|_1)$ to $((E, \| \,.\, \|_2)', *\sigma((E, \| \,.\, \|_2)', E))$.*

Proof. Let x_i and ξ_i be sequences in U and $(E, \| \,.\, \|_2)'$ such that $\xi_i \in \partial_a^2 h_2(x_i)$, x_i converges to x with respect to the norm $\| \,.\, \|_1$, and ξ_i converges to ξ for the weak topology $*\sigma((E, \| \,.\, \|_2)', E)$.

Since $\xi_i \in \partial_a^2 h_2(x_i)$, then there exists $x_{i,r} \rightarrow_{\|.\|_2} x_i$ and $\phi_{r,i} \in \partial_-^2 h_2(x_{i,r})$ such that $\phi_{r,i} \rightarrow \xi_i$ for the weak topology $*\sigma((E, \| \,.\, \|_2)', E)$. Consequently, there exists $y_r \rightarrow_{\|.\|_1} x$ and $\theta_r \in \partial_-^2 h_2(y_r)$ such that $\theta_r \rightarrow \xi$ for the weak topology $*\sigma((E, \| \,.\, \|_2)', E)$. Thus, we complete the proof. □

Proposition 3. $\partial_a^{1,2} h_2(\bar{x}) \subset \partial_0^{1,2} h_2(\bar{x})$.

Proof. Let $\phi \in \partial_a^{1,2} h_2(\bar{x})$. Then there are sequences x_r in U and ϕ_r in $\partial_-^2 h_2(x_r)$ such that x_r converges to $\bar{x}$ with respect to the norm $\| \,.\, \|_1$ and ϕ_r converges to ϕ for the weak topology $*\sigma((E, \| \,.\, \|_2)', E)$. Consequently, the

$\| \,.\, \|_2$-Dini directional derivative $h_2^{-,2}(x_r, v) \geq < \phi_r, v >$ for all v in E. On the other hand $h_2^{-,2}(x_r, v) \leq h_2^{0,2}(x_r, v) \leq h_2^{0,1}(x_r, v)$. Hence, by upper semicontinuity of the Clarke directional derivative $h_2^{0,1}$, we deduce that $< \phi, v > \leq h_2^{0,1}(\bar{x}, v)$. Since v is arbitrary, ϕ belongs to $\partial_0^{1,2} h_2(\bar{x})$. Thus, we complete the proof. □

Let us recall now another result which will be used later.

Lemma 4. *At any point x in E, where $h(x) \neq 0$ we have*

$$\partial_a^2 \| h(.) \| (x) = \partial_a^2 < \| h(x) \|^{-1} h(x), h(.) > (x).$$

Using the Lemma 2, we can now present a convenient test for the metric regularity of a function at point x in terms of the generalized Mordukhovich normal cone $N_S^{a,1,2}(x)$.

Theorem 5. *If a point x lies in S and no nonzero element w of Y satisfies the condition*

$$0 \in \partial_a^{1,2} < w, h(.) > (x) + N_S^{a,1,2}(x)\,,$$

then h is $\| \,.\, \|_1$-metrically regular on S at x.

Proof. Let B' denote the unit ball in $(E, \| \,.\, \|_2)'$. If h is not $\| \,.\, \|_1$-metrically regular on S at x then there are sequences (v_r) in S converging to x with respect to the norm $\| \,.\, \|_1$, (y_r) in Y converging to $h(x)$, and (ε_r) in R_{++} decreasing to zero such that the function $\| h(.) - y_r \| + \varepsilon_r \| \,.\, - v_r \|_1$ is minimized on S at v_r. By exact penalization result we deduce for large enough real L

$$0 \in \partial_a^2(\| h(.) - y_r \| + \varepsilon_r \| \,.\, - v_r \|_1 + L d_S^2(.))(v_r).$$

Consequently, $0 \in \partial_a^2(\| h(.) - y_r \|)(v_r) + c\varepsilon_r B' + L\partial_a^2 d_S^2(v_r)$ for all r, using the Limiting subdifferential sum rule. If we write $\omega_r = \| h(v_r) - y_r \|^{-1} (h(v_r) - y_r)$, we obtain by lemma 4

$$0 \in \partial_a^2 < \omega_r, h(.) > (v_r) + c\varepsilon_r B' + L\partial_a^2 d_S^2(.)(v_r),$$

so there are elements u_r in $\partial_a^2 < \omega_r, h(.) > (v_r)$ and z_r in $L\partial_a^2 d_S^2(.)(v_r)$ such that $\| u_r + z_r \|_2 \to 0$. The sequence (ω_r) is bounded in Y and the sequences u_r, and z_r are bounded in $(E, \| \,.\, \|_2)'$, so by taking subsequences we can assume ω_r approaches some nonzero vector ω, z_r approaches some vector z for the weak topology $*\sigma((E, \| \,.\, \|_2)', E)$ and u_r approaches $-z$ for the weak topology $*\sigma((E, \| \,.\, \|_2)', E)$.

Now, using the sum rule again we observe

$$u_r \in \partial_a^2 < \omega, h(.) > (v_r) + \partial_a^2 < \omega_r - \omega, h(.) > (v_r)$$

for each r. The local Lipschitz constant of the function $< \omega_r - \omega, h(.) >$ tends to zero, so since the pair of multifunctions ($\partial_a^2 < \omega, h(.) >$, $\partial_a^{1,2} < \omega, h(.) >$) is closed from $(U, \| \,.\, \|_1)$ to $((E, \| \,.\, \|_2)', *\sigma((E, \| \,.\, \|_2)', E))$, we deduce $-z \in \partial_a^{1,2} < \omega, h(.) > (x)$. Similary, we see $z \in \partial_a^{1,2} L d_S^2(x) \subset N_S^{a,1,2}(x)$, and this contradicts the assumption of the theorem.

Corollary 6. *If* h *is* $(\| \,.\, \|_1, \| \,.\, \|_2)$*-strictly differentiable at the point* x *in* S *and* $(\nabla h(x)^*)^{-1}(N_S^{a,1,2}(x)) = \{0\}$ *or, in particular,* $\nabla h(x)(T_S^{1,2}(x)) = Y$ *then* h *is* $\| \,.\, \|_1$*-metrically regular on* S *at* x.

Proof. Let us remark first that $N_S^{a,1,2}(x) \subset N_S^{1,2}(x)$ by Proposition 3. Since it is easy to check for any element ω of Y the function $< \omega, h(.) >$ is $(\| \,.\, \|_1, \| \,.\, \|_2)$-strictly differentiable at x with derivative $(\nabla h(x))^* \omega$, the first condition implies the result by Theorem 5 and the fact that $\partial_a^{1,2} < \omega, h(.) > (x) = \{(\nabla h(x))^* \omega\}$. On the other hand, the second condition implies the first, since for any element ω of $(\nabla h(x)^*)^{-1}(N_S^{a,1,2}(x))$ there is an element z of $T_S^{1,2}(x)$ satisfying $\nabla h(x) z = \omega$, and now we deduce

$$\| \omega \|^2 = < \omega, \omega > = < \omega, \nabla h(x) z > = < \nabla h(x))^* \omega, z > \leq 0,$$

so $\omega = 0$. □

Let us give now an example of a non $\| \,.\, \|_1$-strictly differentiable function for which the test given in Corollary 6 works.

Example 7. Let Ω a bounded domain in R^2, $E = W_0^{1,2}(\Omega)$ the Sobolev space with the usual norm $\|.\|_2 = \|.\|_{W_0^{1,2}(\Omega)}$. Let also p and ε such that $0 < \varepsilon < 1$, $\varepsilon + 2 < p < \infty$. Set $\|.\|_1 = \|.\|_{L^p(\Omega)}$. Remark that $(E, \|.\|_2)$ is a Banach separable space.

Since $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ then $(E, \|.\|_1, \|.\|_2)$ is a binormed space such that for some $c > 0$ $\|.\|_1 \leq c \|.\|_2$.

Set $g(u) = | u |^{\varepsilon+2}$ and consider the functional G defined on E by

$$G(x) = \int_\Omega g(x(s)) ds.$$

Then G is $\|.\|_2$-twice Fréchet differentiable at every $x \in E$ and

$$G^{(1)}(x) h = \int_\Omega g'(x(s)) h(s) ds,$$

$$G^{(2)}(x)(h_1, h_2) = \int_\Omega g''(x(s)) h_1(s) h_2(s) ds.$$

We assert that G is $(\|.\|_1, \|.\|_2)$-strictly differentiable at every $x \in E$. Indeed, suppose by the contrary that there exists $x \in E$, $\varepsilon > 0$, $x'_m \in E$, $h_m \in E$ such that $x'_m \to x$ in $(E, \|.\|_1)$, $h_m \to 0$ in $(E, \|.\|_1)$ and $\mid r(x'_m, h_m) \mid > \varepsilon \|h_m\|_2$, where

$$r(x', h) = G(x' + h) - G(x') - G^{(1)}(x)h.$$

Then,

$$\mid r(x'_m, h_m) \mid = \mid \int_\Omega \int_0^1 [g'(x'_m(s) + th_m(s)) - g'(x(s))]h_m(s)dtds|.$$

Let $p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Since $p > \varepsilon + 2$ then $p > p'(\varepsilon + 1)$. Thus, $x'_m \to x$ in $(E, \|.\|_{L^{p'(\varepsilon+1)}(\Omega)})$ and $h_m \to 0$ in $(E, \|.\|_{L^{p'(\varepsilon+1)}(\Omega)})$. Without loss of generality we can suppose that there exists $Z_1 \in L^{p'(\varepsilon+1)}(\Omega)$ such that almost everywhere in Ω

$$\mid x'_m(s) \mid + \mid h_m(s) \mid \leq Z_1(s), \ h_m(s) \to 0, \ x'_m(s) \to x(s).$$

Using the Holder inequality, we deduce that

$$\mid r(x'_m, h_m) \mid \leq [\int_\Omega \int_0^1 |g'(x'_m(s) + th_m(s)) - g'(x(s))|^{p'} dtds]^{\frac{1}{p'}} \|h_m\|_{L^p(\Omega)}.$$

Consequently,

$$\mid r(x'_m, h_m) \mid \leq c[\int_\Omega \int_0^1 |g'(x'_m(s) + th_m(s)) - g'(x(s))|^{p'} dtds]^{\frac{1}{p'}} \|h_m\|_2.$$

But this contradicts the fact that $\mid r(x'_m, h_m) \mid > \varepsilon \|h_m\|_2$ since

$$\int_\Omega \int_0^1 |g'(x'_m(s) + th_m(s)) - g'(x(s))|^{p'} dtds \to 0,$$

by the dominated convergence theorem.

Let us remark that G is not $\|.\|_1$-Fréchet differentiable at any point $x \in E$. Indeed, let $\alpha_m \to \infty$ and $d_m \to \infty$ such that $\mid g(d_m) \mid \geq \alpha_m \mid d_m \mid^p$. By the countable additivity of the Lebesgue measure

$$\exists C > 0 \ \exists \Omega' \subset \Omega \text{ such that } \mu(\Omega') > 0, \ \operatorname{dist}(\Omega', \partial\Omega) > 0 \quad \text{and } \forall s \in \Omega' \ \mid x(s) \mid \leq C.$$

In this case, put $D = \max\{g(u) : \mid u \mid \leq C\} < \infty$. Choose $\Omega_m \subset \Omega'$ such that $\mu(\Omega_m) = \mid d_m \mid^{-p} \mid \alpha_m \mid^{-\frac{1}{2}}$ for large m.

Let h_m defined by

$$h_m(s) = \begin{cases} d_m - x(s), & s \in \Omega_m, \\ 0, & s \in \Omega \setminus \Omega_m. \end{cases}$$

It follows then that $\|h_m\|_1 \to 0$ and

$$\mid G(x+h_m) - G(x) \mid \geq \alpha_m \mid d_m \mid^p \mu(\Omega_m) - D\mu(\Omega_m)$$
$$= \mid \alpha_m \mid^{\frac{1}{2}} - D\mu(\Omega_m) \to +\infty.$$

Let $k_m \in C_0^\infty(\Omega)$ such that $\|k_m - h_m\|_1 \to 0$. Then $\|k_m\|_1 \to 0$, but $G(x+k_m) - G(x) \to \infty$ since the Lebesgue integral is absolutely continuous. Therefore, G is not $\|.\|_1$- Fréchet differentiable at x and consequently, G is not $\|.\|_1$-strictly Fréchet differentiable at x.

So to test the metric regularity of the functional G on a closed set S of $(E, \|.\|_1)$, the classical Theorem 1 cannot be used, but we can apply the Corollary 6 since G is $(\|.\|_1, \|.\|_2)$-strictly differentiable at x. Hence, if S is a closed set in $(E, \|.\|_1)$ such that $(\nabla G(x)^*)^{-1}(N_S^{a,1,2}(x)) = \{0\}$ then G is $\|.\|_1$-metrically regular on S at x.

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