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ON HARDY'S INTEGRALS INEQUALITIES

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1. Introduction

In this paper Hardy's integral inequalities are used to get special forms for solutions of some ordinary first order differential equations.

Definition 1. If $P > 1, f(x)$ is a countinous real-valued function such that $f(x) \geq 0$ for every $0 < x < \infty$, let $R(x) = \frac{1}{x} \int_0^x f(t)dt$, then

$$\int_0^\infty R^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx\,. \tag{1}$$

Inequality (1) is a Hardy's First Integral Inequality of a function of one independent variable, see [1]-[6].

Definition 2. Let $f(x), g(x)$ be two continuos real-valued functions and p, $q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b f(x)g(x)dx \leq \left[\int_a^b f^p(x)dx\right]^{\frac{1}{p}} \left[\int_a^b g^p(x)dx\right]^{\frac{1}{q}}. \tag{2}$$

Inequality (2) is a Hölder's Integral Inequality.

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Theorem 1. *Suppose that $P > 1$, $f(x)$ is a real valued function such that $f(x) \geq 0$ $\forall x \in (0, \infty)$, consider the function*

$$R(x) = \frac{1}{a_1 x + b_1} \int_0^x f(x) dx\,,$$

where a_1, b_1 are constants, then

$$\int_0^\infty R^p(x) dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \left[\frac{x\ f(x)}{a_1 x + b_1} + \frac{b_1 \int_a^x f(x) dx}{(a_1 x + b_1)^2}\right]^p dx\,.$$

Proof. Let

$$R(x) = \frac{1}{a_1 x + b_1} \int_a^x f(x) dx\,, \tag{3}$$

where a is a real arbitrary constant. Integrating the function $R^p(x)$ from a to b, we get

$$\int_a^b R^p(x) dx = [x\ R^p(x)]_a^b - p \int_a^b x\, R^{p-1}(x).\ R^{'}(x) dx\,.$$

Differentiating (3) with respect to x and let $a = 0$ we get

$$\int_a^b R^p(x) dx \leq \frac{p}{p-1} \int_0^b \left[R^{p-1}(x) \left(\frac{x f(x) + b_1 R(x)}{a_1 x + b_1}\right)\right] dx\,. \tag{4}$$

Applying Hölder's Integral Inequality to the right sinde of inequality (4) with indices p, $\frac{p}{p-1}$, where $\frac{1}{p} + 1\ /\ \left(\frac{p}{p-1}\right) = 1$, then

$$\int_0^b R^p(x) dx \leq \left(\frac{p}{p-1}\right)^p \int_0^b \left[\frac{x f(x) + bR(x)}{a_1 x + b_1}\right]^p dx\,.$$

Let b tends to ∞, we get

$$\int_0^\infty R^p(x) dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \left[\frac{x f(x) + b_1 R(x)}{a_1 x + b_1}\right]^p dx \tag{5}$$

Also from (3) we get

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \left[\frac{xf(x)}{a_1x+b_1} + \frac{b_1\int_0^x f(x)dx}{(a_1x+b_1)^2}\right]^p dx\,. \qquad (6)$$

From above theorem we can deduce that:

1) If $a_1 = 1, b = 0$, the mean results, (5) and (6) gives Hardy's integral inequality.

2) Function (3) is a solution of the first order linear differential equation

$$(a_1x+b_1)\frac{dR(x)}{dx} + a_1R(x) = f(x)\,.$$

3) If $f(x) = c$, where c is an arbitrary constant, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \sum_{j=1}^{2}\left[\frac{cb_1^{j-1}}{(a_1x+b_1)^j}\right]^p x^p dx\,.$$

4) If $f(x) = x^{n-1}$ we get

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty x^{np}\left[\sum_{j=1}^{2}\frac{b_1^{j-1}}{n^{j-1}(a_1x+b_1)^j}\right]^p dx\,.$$

5) If $f(x) = \sin x$, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \left[\frac{x\sin x}{(a_1x+b_1)} + \frac{1-\cos x}{(a_1x+b_1)^2}\right]^p dx\,.$$

6) If $f(x) = e^x$, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \left[\frac{e^x(a_1x^2+b_1x+b_1)-b_1}{(a_1x+b_1)^2}\right]^p dx$$

Theorem 2. *Suppose that $P > 1, f(x)$ is a real valued function, such that $f(x) \ge 0$ for every $x \in (0,\infty)$, consider the function*

$$R(x) = \frac{1}{2(a_1x+b_1)^2}\int_0^x f(x)dx \quad ,$$

where a_1, b_1 are arbitrary constants, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^\infty \left[\frac{x\,f(x)}{2(a_1x+b_1)^2} + \frac{b_1\int_0^x f(x)dx}{(a_1x+b_1)^3}\right]^p dx\,.$$

Proof. Since

$$R(x) = \frac{1}{2(a_1x+b_1)^2}\int_0^x f(x)dx \tag{7}$$

and

$$\int_a^b R^p(x) = [xR^p(x)]_a^b - p\int_a^b xR^{'}(x)\,R^{p-1}(x)dx\,, \tag{8}$$

differentiating (7) with respect to x and let $a = 0$, then from (8) we get

$$\int_0^b R^p(x)dx \le \left(\frac{p}{2p-1}\right)\left[\int_0^b R^{p-1}(x)\left(\frac{x\,f(x)}{2(a_1x+b_1)^2} + \frac{2b_1R(x)}{a_1x+b_1}\right)\right] dx.$$

Applying Hölder's Integral Inequality to the right said of the above inequality with indices p, $\frac{p}{p-1}$, we get

$$\int_0^b R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^b \left[\frac{x\,f(x)}{2(a_1x+b_1)^2} + \frac{2b_1R(x)}{a_1x+b_1}\right]^p dx\,. \tag{9}$$

Let $b \to \infty$ we have

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^\infty \left[\frac{x\,f(x)}{2(a_1x+b_1)^2} + \frac{2b_1R(x)}{(a_1x+b_1)^3}\right]^p dx\,. \tag{10}$$

Using (7) we get

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^\infty \left[\frac{x\,f(x)}{2(a_1x+b_1)^2} + \frac{b_1\int_0^x f(x)dx}{(a_1x+b_1)^3}\right]^p dx\,. \tag{11}$$

We can obsirve that:

1) If $a_1 = 1, b_1 = 0$ we get Hardy's integral inequality.

2) The function (7) is a solution of first order linear differential equation

$$(a_1x+b_1)^2\frac{dR(x)}{dx}+2a_1(a_1x+b_1)R(x)=\frac{1}{2}f(x)\,.$$

Theorem 3. *Suppose that $P > 1, f(x)$ is a real valued function, such that $f(x) \geq 0$ for every $x \in (0,\infty)$, consider the function*

$$R(x)=\frac{1}{n(a_1x+b_1)^n}\int_0^x f(x)dx,$$

then

$$\int_0^\infty R^p(x)dx \leq \left(\frac{p}{np-1}\right)^p \int_0^\infty \left[\frac{x\,f(x)}{n(a_1x+b_1)^n}+\frac{b_1\int_0^x f(x)dx}{(a_1x+b_1)^{n+1}}\right]^p dx.$$

Proof. Since

$$\int_a^b R^p(x)=[xR^p(x)]_a^b - p\int_a^b xR^{p-1}(x)\,R'(x)dx \tag{12}$$

and

$$R(x)=\frac{1}{n(a_1x+b_1)^n}\int_0^x f(x)dx, \tag{13}$$

then

$$R'(x)=\frac{f(x)}{n(a_1x+b_1)^n}-\frac{a_1nR(x)}{(a_1x+b_1)}.$$

Let $a = 0$, then (12) becomes

$$\int_0^b R^p(x)dx$$

$$\leq \left(\frac{p}{np-1}\right)\left[\int_0^b R^{p-1}(x)\left(\frac{x\,f(x)}{n(a_1x+b_1)^n}+\frac{nb_1R(x)}{a_1x+b_1}\right)dx\right]. \tag{14}$$

Applying Hölder's Integral Ineqality to (14) with indices $p, \frac{p}{p-1}$, we get

$$\int_0^b R^p(x)dx \le \left(\frac{p}{np-1}\right)^p \int_0^b \left(\frac{x\,f(x)}{n(a_1x+b_1)^n} + \frac{nb_1R(x)}{a_1x+b_1}\right)^p dx\,.$$

Let $b \to \infty$ we get

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{np-1}\right)^p \int_0^\infty \left(\frac{x\,f(x)}{n(a_1x+b_1)^n} + \frac{nb_1R(x)}{a_1x+b_1}\right)^p dx\,,$$

also

$$\int_0^\infty R^p(x)dx$$

$$\le \left(\frac{p}{np-1}\right)^p \int_0^\infty \left(\frac{x\,f(x)}{n(a_1x+b_1)^n} + \frac{b_1\int_0^x f(x)dx}{(a_1x+b_1)^{n+1}}\right)^p dx\,. \quad (15)$$

We can observe that:

1) If $a_1 = \frac{1}{n}$, $b_1 = 0$, $n = 1$ we get Hardy's integral inequality.

2) Function (13) is a solution of the first order linear differential equation

$$R^{'}(x) + \frac{a_1 n}{(a_1x+b_1)}\; R(x) = \frac{f(x)}{n(a_1x+b_1)^n}\,.$$

3) If $f(x) = c$, where c is an arbitrary constant, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{np-1}\right)^p \int_0^\infty \sum_{j=1}^{2} \left[\frac{x\;\; cb_1^{j-1}}{n^{2-j}(a_1x+b_1)^{n+j-1}}\right]^p dx.$$

4) If $f(x) = x^{n-1}$ we have

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{np-1}\right)^p \int_0^b \left[\frac{x^n}{n}\sum_{j=1}^{2} \frac{b_1^{j-1}}{(a_1x+b_1)^{n+j-1}}\right]^p dx.$$

5) If $f(x) = e^x$, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{np-1}\right)^p \int_0^b \left[\frac{xe^x}{n(a_1x+b_1)^n} + \frac{b_1(e^x-1)}{(a_1x+b_1)^{n+1}}\right]^p dx$$
$$\le \left(\frac{p}{np-1}\right)^p \int_0^\infty \left(\frac{e^x(a_1x^2+b_1x+nb_1)-nb_1}{(a_1x+b_1)^{n+1}}\right)^p dx\,.$$

Theorem 4. *Suppose that $p > 1, f(x)$ is a real valued function, such that $f(x) \ge 0 \;\forall\, x \in (0,\infty)$, consider the function*

$$R(x) = \frac{1}{a_1x^2+b_1}\int_0^x f(x)dx,$$

then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^\infty \left[\frac{x\,f(x)}{a_1x^2+b_1} + \frac{2b_1\int_0^x f(x)dx}{(a_1x^2+b_1)^2}\right] dx.$$

Proof. Since

$$R(x) = \frac{1}{a_1x^2+b_1}\int_0^x f(x)dx, \tag{16}$$

then

$$R^{'}(x) = \frac{1}{a_1x^2+b_1}f(x) - \frac{2a_1x}{a_1x^2+b_1}R(x). \tag{17}$$

Also

$$\int_a^b R^p(x) = [xR^p(x)]_a^b - p\int_a^b xR^{p-1}(x)\,R'(x)dx.$$

Let $a = 0$, we get

$$\int_0^b R^p(x)dx \le \frac{p}{2p-1}\int_0^b \frac{xf(x)+2b_1R(x)}{a_1x^2+b_1}R^{p-1}(x)dx, \tag{18}$$

applying Hölder's Integral Inequality to inequality (18) by indices p and $\frac{p}{p-1}$, we get

$$\int_0^b R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^b \left[\frac{x\,f(x)}{a_1x^2+b_1} + \frac{2b_1R(x)}{a_1x^2+b_1}\right]^p dx\,.$$

Let $b \to \infty$, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^\infty \left[\frac{x\,f(x)}{a_1x^2+b_1} + \frac{2b_1R(x)}{a_1x^2+b_1}\right]^p dx\,,$$

also

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^\infty \left[\frac{x\,f(x)}{(a_1x^2+b_1)} + \frac{2b_1\int_0^x f(x)dx}{(a_1x^2+b_1)^2}\right]^p dx\,. \tag{19}$$

We can observe that:

1) If $b=0, a_1=1$ we get Hardy's integral inequality.

2) $R(x) = \frac{1}{a_1x^2+b_1}\int_0^x f(x)dx$ is a solution of the first order linear differential equation

$$R'(x) + \frac{2a_1x}{(a_1x^2+b_1)}R(x) = \frac{1}{a_1x^2+b_1}f(x).$$

3) If $f(x)=c,\ c$ is an arbitrary constant, then

$$\int_0^\infty R^p(x)dx \le \left(\frac{p}{2p-1}\right)^p \int_0^\infty \left[\sum_{j=1}^{2} cx\left[\frac{2^{j-1}\ b_1^{j-1}}{(a_1x^2+b_1)^j}\right]\right]^p dx\,.$$

References

[1] M. Alic, J. Pecaric, Inequalities related to Hardy's and Leyin's, *Rad. Hrvatske. Akad. Znan. Umj. Mat.*, **64711** (1974), 111-117.

[2] E.F. Beckenbach, R. Bellman, *Inequalities*, Springer Verlag, Berlin-Heidelberg-New York (1970).

[3] R.P. Boas, Jr., Some integral inequalities related to Hardy's inequality, *J. Analyse. Math.*, **23** (1970), 53-63.

[4] G.H. Hardy, Notes on a theorem of Hilbert, *Math. Z.*, **6** (1920), 314-317.

[5] G.H. Hardy, Notes on some points in the integral calculus, *Messenger of Math.*, **57** (1928), 12-16.

[6] Lzumim, S. Izumi, C. Peterson, On Hardy's inequality and its generalizations, *Tohoku. Math. J.*, **21** (1996), 601-613.