International Journal of Pure and Applied Mathematics
Volume 20 No. 3 2005, 299-302

GROUPS WITH TRIVIAL INTERSECTION OF SUBGROUPS

Hamza Ahmad[1 §], Nancy Colwell[2]
[1,2]Department of Mathematical Sciences
Saginaw Valley State University
7400 Bay Road, University Center, MI 48710, USA
[1]e-mail: nccolwel@svsu.edu
[2]e-mail: hyahmad@svsu.edu

Abstract: We determine, up to isomorphism, all finite groups G that have the property that for any two subgroups H and K of G, $H \subseteq K$, $K \subseteq H$, or $|H \cap K| = 1$.

AMS Subject Classification: 20E99, 20D20
Key Words: finite group, trivial intersection, subgroup

1. Introduction

In this note we determine all finite groups G that have the property

$$\begin{cases} \text{For any two subgroups } H \text{ and } K \text{ of } G, \text{ we have } H \subseteq K,\ K \subseteq H, \text{ or} \\ H \cap K = \{e\}, \text{ where } e \text{ is the identity of } G. \end{cases} \qquad (*)$$

It is easy to verify that the following list of groups satisfy the above property. (1) G is cyclic of order p^m, (2) G has order p^2 or pq, and (3) G has order p^2q with no subgroup of order pq, where p and q are distinct primes. We will show that this list is actually complete.

Our arguments are elementary. Basic background and standard notation can be found in any group theory book (e.g. [1]). In particular, $|G|$ denotes the order of G, for $x \in G$, $\langle x \rangle$ is the subgroup generated by x, $H \triangleleft G$ means H is

Received: January 31, 2005
§Correspondence author

a normal subgroup of G, and $\mathcal{N}_G(H)$ denotes the normalizer of H in G. Note that if G is Abelian and satisfies $(*)$, it follows from the fundamental theorem of finite Abelian groups that G must be a cyclic p-group or of order pq, or G has order p^2.

2. The Results

Proposition 1. *Suppose that G is a p-group satisfying $(*)$. Then G is cyclic or of order p^2.*

Proof. Suppose that $|G| = p^m$, where $m > 2$. Suppose that H and K are two distinct subgroups of order p^{m-1}. Then H and K are normal and $HK = G$. Since $p^m = |G| = |HK| = |H|\,|K|\,/\,|H \cap K| = p^{2m-2}/\,|H \cap K|$, we have $|H \cap K| = p^{m-2} > 1$; a contradiction. So G has a unique subgroup of order p^{m-1}. Since every element of order p^i, with $i < m$, is contained in this unique subgroup of order p^{m-1}, it follows that G must have an element of order p^m; i.e. G must be cyclic. □

Lemma 2. *Let G satisfy $(*)$ and G is not a p-group. Suppose that for a prime p dividing $|G|$, the Sylow p-subgroup P of G is normal. Then $|P| = p$ or p^2.*

Proof. Suppose that $|P| = p^m$, where $m > 2$. P satisfies $(*)$; hence, by the previous proposition, P is cyclic. Therefore G contains a unique subgroup H of order p^{m-1}. In particular $H \lhd G$. Since G is not a p-group, it contains a Sylow q-subgroup Q, where $q \neq p$. Therefore the set HQ is a subgroup of order $p^{m-1}\,|Q|$. Hence $P \cap HQ = H \neq \{e\}$; a contradiction. □

Lemma 3. *If G satisfies $(*)$ and, for a prime p dividing $|G|$, the Sylow p-subgroup P of G is normal, then:*

1. *G is p-group which is cyclic or has order p^2, or*
2. *$|G| = pq$ or p^2q (for some prime $q \neq p$).*

Proof. If G is a p-group, then G is of type (1) by Proposition 1. So assume that G is not a p-group. By Lemma 2, $|P| = p$ or p^2. Suppose $q \neq p$ is another prime factor of $|G|$, and let Q be a Sylow q-subgroup of G of order q^m. If $m \geq 2$, Q contains a proper non-trivial subgroup Q_0 of order q^{m-1}. Since P is normal, PQ_0 is a subgroup, and $Q \cap PQ_0 = Q_0 \neq \{e\}$; a contradiction. Therefore $|Q| = q$. It is left to show that $|G|$ is not divisible by a prime other than p and q. Assume that $r \neq p, q$ is a prime factor of $|G|$, and R is a Sylow r-subgroup.

Since P is normal, PQ and PR are subgroups of G of orders $q\,|P|$ and $r\,|P|$, respectively. Note that $PQ \cap PR = P \neq \{e\}$; a contradiction. $\square$

Now we will show that a group that satisfies $(*)$ contains a normal Sylow subgroup.

Lemma 4. *Let G be a group satisfying $(*)$. Then there exist Sylow subgroups $P_1, \ldots, P_r$ such that:*

1. $|\mathcal{N}_G(P_1)|, \ldots, |\mathcal{N}_G(P_r)|$ are relatively prime.

2. $|G| = \prod_{i=1}^{r} |\mathcal{N}_G(P_i)|$.

3. For any $x \in G - \{e\}$, there exists a unique i such that x belongs to a conjugate of $\mathcal{N}_G(P_i)$ (which is necessarily unique by the property $(*)$).

Proof. For a prime p dividing $|G|$, let P be a Sylow p-subgroup of G and set $N = \mathcal{N}_G(P)$. Suppose $q \neq p$ is a prime factor of $|N|$ and H is a Sylow q-subgroup of N. Then H is contained in a Sylow q-subgroup Q of G. Note that $N \not\subseteq Q$ and $Q \cap N = H \neq \{e\}$. Therefore, by $(*)$, we have $Q \subset N$. Hence, if q is any prime dividing $|N|$, then q does not divide $|G|\,/\,|N|$.

Now we can choose the P_i's as follows: Among all the Sylow subgroups, pick P_1 such that $|\mathcal{N}_G(P_1)|$ is maximal. Recursively choose P_i so that $|P_i|$ is coprime with $|\mathcal{N}_G(P_1)|, \ldots, |\mathcal{N}_G(P_{i-1})|$ and $|\mathcal{N}_G(P_i)|$ is maximal. The sequence of Sylow subgroups chosen this way satisfies (1) and (2). To see that (3) holds, let $x \in G$ and let q be a prime divisor of $|x|$. Then $\langle x \rangle$ contains a non-trivial subgroup H of order q. H is contained in a Sylow q-subgroup Q. By (1) and (2), there is a unique i such that q divides $|\mathcal{N}_G(P_i)|$. By the first paragraph of the proof, $\mathcal{N}_G(P_i)$ contains a Sylow q-subgroup of G. Since the Sylow q-subgroups are conjugate, Q is a subset of a conjugate N' of $\mathcal{N}_G(P_i)$. Without loss of generality, we may assume $N' = \mathcal{N}_G(P_i)$. If $\mathcal{N}_G(P_i) \subseteq \langle x \rangle$, then x commutes with all elements of P_i; hence $\langle x \rangle \subseteq \mathcal{N}_G(P_i)$. Now $\langle x \rangle \cap \mathcal{N}_G(P_i) \supset H \neq \{e\}$. Therefore, by $(*)$, we get $x \in \mathcal{N}_G(P_i)$. $\square$

Proposition 5. *If G satisfies $(*)$, then G has a normal Sylow subgroup.*

Proof. This is trivially true for a p-group. So assume that G is not a p-group. Let $P_1, \ldots, P_r$ be the Sylow subgroups as in the previous lemma. We shall show that $r = 1$ and therefore $P_1 \lhd G$. Let $n = |G|$, $n_i = |\mathcal{N}_G(P_i)|$, and ν_i be the number of conjugates of $\mathcal{N}_G(P_i)$. By [1, Corollary 4.33, p. 83], $\nu_i = n/n_i$. By (3) of Lemma 4, we have $n - 1 = \sum_{i=1}^{r} \nu_i(n_i - 1) = \sum_{i=1}^{r} n(n_i - 1)/n_i$, or equivalently

$$\frac{n-1}{n} = \sum_{i=1}^{r} \frac{n_i - 1}{n_i}\,.$$

Note that $(n-1)/n < 1$. Since $n_i \geq 2$, $(n_i - 1)/n_i \geq 1/2$. So the only way to satisfy the displayed equality is to have $r = 1$, i.e. $n_1 = n$ and P_1 is normal in G. □

By Proposition 5 and Lemma 3 we get the following corollary.

Corollary 6. *G satisfies $(*)$ if and only if:*
1. G is a cyclic p-group,
2. $|G| = p^2$ or pq, or
3. $|G| = p^2q$ and G has no subgroup of order pq,
where q and p are distinct primes.

The group of type (3) in the above corollary can be further described as follows. The Sylow p-subgroup P is normal of order p^2. Let Q be a Sylow q-subgroup. P cannot be cyclic, otherwise P (and therefore G) will have a unique subgroup H of order p. In particular, $H \triangleleft G$. Thus HQ is a subgroup of G, and we have $HQ \cap P = H \neq \{e\}$; a contradiction. Since P is Abelian, we have $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$. As G does not have a subgroup of order pq, it follows that G is the semi-direct product of $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_q (\cong Q)$.

References

[1] John Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge (1978).