International Journal of Pure and Applied Mathematics
Volume 20 No. 3 2005, 329-332

NOTES ON HUPPERT'S ρ-σ-CONJECTURE

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Abstract: For a finite group G, in this note, we prove that if G is nilpotent-by-nilpotent, then Huppert's ρ-σ-conjecture is valid for G. Also we show that if $G/F(G)$ is nilpotent and Sylow subgroups of $F(G)$ are all non-Abelian, then the conjugacy version of Huppert's ρ-σ-conjecture is true for G.

AMS Subject Classification: 20C15
Key Words: finite group, conjugacy class, Sylow subgroup

1. Preliminaries

Let G be a finite group, $\pi(n)$ be the set of different prime divisors of the positive integer n. Write $Cl(g)$ to denote the set of all conjugates of g in G and $Irr(G)$ to denote the set of the irreducible complex characters of G. Set

$$\rho(G) = \bigcup_{\chi \in Irr(G)} \pi(\chi(1)), \qquad \sigma(G) = \operatorname*{Max}_{\chi \in Irr(G)} \{|\pi(\chi(1))|\},$$

$$c\rho(G) = \bigcup_{g \in G} \pi(|Cl(g)|), \qquad c\sigma(G) = \operatorname*{Max}_{g \in G} \{|\pi(|Cl(g)|)|\}.$$

For a solvable group G, B. Huppert [3, 1] conjectured that $|\rho(G)| \le 2\sigma(G)$. Since there is some analogy between conjugacy lengths and character degrees of a finite group, it was also conjectured that $|c\rho(G)| \le 2c\sigma(G)$.

In this note, we discuss the conjectures. Note that there exist supersolvable and metabelian groups which are counterexamples against the latter conjecture.

Received: March 5, 2005

Lemma 1.1. *Assume that $N \lhd\lhd G, M \trianglelefteq G$. Then:*
1. $\rho(N) \subseteq \rho(G)$, $\sigma(N) \leq \sigma(G)$;
2. $\rho(G/M) \subseteq \rho(G)$, $\sigma(G/M) \leq \sigma(G)$.

Proof. Omitted. □

Lemma 1.2. *Let G be solvable and $F(G)$ be Fitting subgroup of G. Set $\mathcal{T}(G) = \{p|\ O_p(G)$ is non-Abelian$\}$, then $\rho(G) = \pi(G/F(G)) \cup \mathcal{T}(G)$.*

Proof. See the proof of [4, Theorem 17.7]. □

Observe that if $\mathcal{T}(G)$ were nontrivial, then it should easily find $\chi \in$Irr(G) such that $\pi(\chi(1)) \supseteq \mathcal{S}(G)$. Thus the problem probably reduces to the investigation of $\pi(G/F(G))$.

2. Main Results

Theorem 2.1. *Suppose that G is nilpotent-by-nilpotent (metanilpotent), then $|\rho(G)| \leq 2\sigma(G)$.*

Proof. Assume that $G/F(G)$ is an Abelian group. By Gaschütz's Theorem 1.12 and Proposition 12.1 of [4], it follows that $\mathrm{Irr}\,(F(G)/\Phi(G))$ is a faithful and completely reducible $G/F(G)$-module, let

$$\mathrm{Irr}\,(F(G)/\Phi(G)) = V_1 \oplus V_2 \oplus \cdots \oplus V_n,$$

V_i's are faithful and irreducible $\bar{G}/C_{\bar{G}}(V_i)$-modules respectively, where $\bar{G} = G/F(G)$ and $C_{\bar{G}}(V_i) = \{v \in V_i|\ vg = v \text{ for any } g \in \bar{G}\}$. Because $C_{\bar{G}}(V_i) \geq F(G)$ and $G/F(G)$ is Abelian, we get that $\bar{G}/C_{\bar{G}}(V_i)$ acts fixed-point-free on V_i. Set $\eta = \lambda_1\lambda_2\cdots\lambda_n$, then

$$I_G(\eta) = \cap_{i=1}^n I_G(\lambda_i) = \cap_{i=1}^n C_G(V_i) = F(G),$$

we get by Clifford's Theorem 6.11 of [2] that $\chi = \eta^G$ is irreducible. Let $\theta \in \mathrm{Irr}\,(F(G))$ faithful, and $\xi \in \mathrm{Irr}\,(G|\theta)$, then $\rho(G) = \pi(\chi(1)) \cup \pi(\xi(1))$. Hence

$$|\rho(G)| = |\pi(\chi(1)) \cup \pi(\xi(1))| \leq 2\max\{|\pi(\chi(1))|, |\pi(\xi(1))|\} \leq 2\sigma(G).$$

By the hypothesis, we know that $G/F(G)$ is nilpotent. We may assume that $G/F(G)$ is Abelian. For the nilpotent group $G/F(G)$, we see that $\pi(G/F(G)) = \pi(Z(G/F(G)))$. Set $Z/F(G) = Z(G/F(G))$, then since $F(G) \leq F(Z)$, and $F(Z)\,\mathrm{char}\,Z \trianglelefteq G$ so that $F(Z) \leq F(G)$, it follows that $F(G) = F(Z)$, thus $F(G)/\Phi(G)$ is a faithful and completely reducible $Z/F(G)$-module, by the above, it implies that $|\rho(Z)| \leq 2\sigma(Z)$.

By Lemma 1.2, we know that $|\rho(G)| = |\pi(G/F(G)) \cup \mathcal{T}(F(G))|$ and $|\rho(Z)| = |\pi(Z/F(Z)) \cup \mathcal{T}(F(Z))|$. Also since $F(G) = F(Z)$, we conclude that

$$|\rho(G)| = |\rho(Z)| \leq 2\sigma(Z) \leq 2\sigma(G),$$

the proof is of complete. □

The following result is easily obtained from the above theorem.

Corollary 2.2. *Suppose that the finite group G is supersolvable, then $|\rho(G)| \leq 2\sigma(G)$.*

Theorem 2.3. *Suppose that $G/F(G)$ is nilpotent, and Sylow subgroups of $F(G)$ are all non-Abelian, then $|c\rho(G)| \leq 2c\sigma(G)$.*

Proof. Assume that $G/F(G)$ is Abelian. By Gaschütz's Theorem 1.12 of [4], it follows that $F(G)/\Phi(G)$ is a faithful and completely reducible $G/F(G)$-module, let

$$F(G)/\Phi(G) = V_1 \oplus V_2 \oplus \cdots \oplus V_n,$$

V_i's are faithful and irreducible $\bar{G}/C_{\bar{G}}(V_i)$-modules respectively, where $\bar{G} = G/F(G)$ and $C_{\bar{G}}(V_i) = \{v \in V_i |\ vg = v \text{ for any } g \in \bar{G}\}$. Because $C_G(V_i) \geq F(G)$ and $G/F(G)$ is Abelian, we get that $\bar{G}/C_{\bar{G}}(V_i)$ acts fixed-point-free on V_i. Set $v = v_1 v_2 \cdots v_n$, then

$$C_G(v) = \cap_{i=1}^{n} C_G(v_i) = \cap_{i=1}^{n} C_G(V_i) = F(G),$$

we get by the above that $|Cl_G(v)| = |G : F(G)|$. Because $F(G)$ is nilpotent and none of its Sylow subgroups are Abelian, we may choose $w \in F$ such that $\pi(|Cl_{F(G)}(w)|) = \pi(F(G))$; further we may see that $\pi(|Cl_G(w)|) \supseteq \pi(F(G))$ and $\pi(G) = \pi(|Cl_G(w)|) \cup \pi(|Cl_G(v)|)$. Hence

$$\begin{aligned} |c\rho(G)| &= |\pi(|Cl_G(v)|) \cup \pi(|Cl_G(w)|)| \\ &\leq 2\max\{|\pi(|Cl_G(v)|)|, |\pi(|Cl_G(w)|)|\} \leq 2c\sigma(G). \end{aligned}$$

By the hypothesis, we know that $G/F(G)$ is nilpotent. We may assume that $G/F(G)$ is Abelian. For the nilpotent group $G/F(G)$, we see that $\pi(G/F(G)) = \pi(Z(G/F(G)))$. Set $Z/F(G) = Z(G/F(G))$, then since $F(G) \leq F(Z)$, and $F(Z) \operatorname{char} Z \trianglelefteq G$ so that $F(Z) \leq F(G)$, it follows that $F(G) = F(Z)$, thus $F(G)/\Phi(G)$ is a faithful and completely reducible $Z/F(G)$-module, by the above, it implies that $|\pi(Z)| = |c\rho(Z)| \leq 2c\sigma(Z)$, hence we get that

$$|c\rho(G)| \leq |\pi(G)| = |\pi(Z)| = |c\rho(Z)| \leq 2c\sigma(Z) \leq 2c\sigma(G),$$

as required. □

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