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COHERENT SYSTEMS WITH MANY "SPREAD" SECTIONS ON CURVES

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Abstract: Let X be a genus g hyperelliptic projective curve, $\alpha \in \mathbb{R}$, $\alpha \geq 0$, and integers $n \geq 2$, k and d such that $0 \leq d \neq n(2g-2)$. Here we prove when there is an α-stable coherent system (E, V) on X of type (n, d, k) such that both E and $\omega_X \otimes E^*$ are spanned if and only if d is even and there is an $(\alpha/2)$-stable coherent system of type $(n, d/2, k)$ on $\mathbf{P}^1$.

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1. Primitive Coherent Systems

For the general theory of coherent systems on a smooth curve, see and references therein. For any smooth and connected projective curve C, let $G(C; \alpha; n, d, k)$ denote the moduli spaces of all α-stable coherent system on C of type (n, d, k). For a study of coherent systems on $\mathbf{P}^1$, see [2]. Here we will consider the case of a hyperelliptic curve X of genus $g \geq 2$. However, we will only study coherent systems (E, V) on X for which both E and $\omega_X \otimes E^*$ is spanned; we will not require that V spans. Following [1] we will say that a vector bundle F on a smooth and connected projective curve C is *primitive* if both F and $\omega_C \otimes F^*$ are spanned. Set $G(C; \alpha; n, d, k)^+ := \{(E, V) \in G(C; \alpha; n, d, k) : E \text{ is primitive}\}$. Here we will prove the following result.

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Theorem 1. *Let X be a smooth hyperelliptic curve of genus $g \geq 2$, $R \in \mathrm{Pic}^2(X)$ the hyperelliptic line bundle and $f : X \to \mathbf{P}^1$ the associated degree 2 covering. Then:*

(i) $G(X;\alpha;n,d,k)^+ \neq \emptyset$ if and only d is even, $G(\mathbf{P}^1;\alpha/2;n,d/2,k) \neq \emptyset$ and $0 \leq d \leq n(2g-2)$. If $G(X;\alpha;n,d,k)^+ \neq \emptyset$, then it is an irreducible locally closed subset of $G(X;\alpha;n,d,k)$.

(ii) Assume $G(X;\alpha;n,d,k)^+ \neq \emptyset$; then there is

$$(F,W) \in G(\mathbf{P}^1;\alpha/2;n,d/2,k)$$

such that

$$(f^*(F), f^*(W)) \in G(\mathbf{P}^1;\alpha/2;n,d/2,k).$$

(iii) Assume $G(X;\alpha;n,d,k)^+ \neq \emptyset$ and set $a := \lfloor d/2n \rfloor$, $b := d/2 - na$. Then there is $(E,V) \in G(X;\alpha;n,d,k)^+$ such that

$$E \cong (R^{\otimes a})^{\oplus(n-b)} \oplus (R^{\otimes(a+1)})^{\oplus b}.$$

Proof. Let $S(n,d)$ denote the set of all primitive vector bundles on X with rank n and degree d. Let $S'(n,x)$ denote the set of all spanned vector bundles on $\mathbf{P}^1$ with rank n and degree x. By [1], Theorem 1, $E \in S(n,d)$ if and only if there are integers $g-1 \geq a_1 \geq \cdots \geq a_n \geq 0$ such that $E \cong \oplus_{i=1}^n R^{\otimes a_i}$. Hence $S(n,d) \neq \emptyset$ if and only if d is even and $0 \leq d \leq n(2g-2)$. Furthermore, for each $E \in S(n,d)$ there is $F \in S'(n,d/2)$ such that $E \cong f^*(F))$ and $H^0(X,E) = f^*(H^0(\mathbf{P}^1,F))$. Furthermore, both sets $S(n,d)$ and $S'(n,d/2)$ appear in the deformation space of one of their members and these deformation spaces (call them $D(n,d)$ and $D'(n,d/2)$) are irreducible. The general member of $D'(n,d/2)$ is the rigid bundle, i.e. (with the notation of part (iii) of Theorem 1) the vector bundle $G := \mathcal{O}_{\mathbf{P}^1}(a)^{\oplus(n-b)} \oplus \mathcal{O}_{\mathbf{P}^1}(a+1)^{\oplus b}$. By [2], Theorem 3.2, if $G(\mathbf{P}^1;\alpha/2;n,d/2,k) \neq \emptyset$, then there is a k-dimensional linear subspace M of $H^0(\mathbf{P}^1,G)$ such that $(G,M) \in G(\mathbf{P}^1;\alpha/2;n,d/2,k)$. Furthermore, $(G,W) \in G(\mathbf{P}^1;\alpha/2;n,d/2,k)$ for a general k-dimensional linear subspace W of $H^0(\mathbf{P}^1,G)$. By the openness of α-stability and the irreducibility of $D(n,d)$, we get the "furthermore part" of assertion (i) and (since $h^0(X,f^*(G)) = h^0(\mathbf{P}^1,G)$) that if $G(X;\alpha;n,d,k)^+ \neq \emptyset$, then a general element of $G(X;\alpha;n,d,k)^+$ is of the form $(f^*(G), f^*(W)$ with W general k-dimensional linear subspace W of $H^0(\mathbf{P}^1,G)$. To prove all the assertions of the theorem it is sufficient to prove that if $G(\mathbf{P}^1;\alpha/2;n,d/2,k) \neq \emptyset$, then $(f^*(G), f^*(W)) \in G(X;\alpha;n,d,k)^+$. Since $f^*(G)$ is primitive, by the inequality $d \leq n(2g-2)$ (which implies either $a+1 \leq g-1$ or $a = g-1$, $b = 0$

and $E \cong \omega_X^{\oplus n}$), it is sufficient to prove that $(f^*(G), f^*(W))$ is α-stable. Assume that $(f^*(G), f^*(W))$ is not α-stable and take a proper coherent subsystem (T, B) of $(f^*(G), f^*(W))$ with $\mu_\alpha(T, B) \geq \mu_\alpha(f^*(G)f^*(W)) = \alpha k/n + d/n$ and with maximal α-slope. Hence $B = H^0(X, T) \cap f^*(W)$. We also assume that rank(T) is maximal among such coerent subsystems with the same α-slope. By the maximality of $\mu_\alpha(T, B)$ the coherent system (T, B) is α-semistable. There is a unique $B' \subseteq W$ such that $B = f^*(B')$. We have $B = H^0(X, B) \cap f^*(W)$. Let $\sigma : X \to X$ the hyperelliptic involution. Since $f^*(G)$ comes from $\mathbf{P}^1$, σ acts on the set of all subsheaves of $f^*(G)$. Since $f^*(W)$ comes from $\mathbf{P}^1$, σ acts as the identity on $f^*(W)$. Hence $\sigma^*((T, B)) = (\sigma^*(T), B)$. Since (T, B) is α-semistable, $(T, B) \oplus (\sigma^*(T), B)$ is α-semistable and hence its image $(T + \sigma^*(T), B)$ in $(f^*(G), f^*(W))$ has α-slope $\geq \mu_\alpha(T, B)$. By the maximality of rank(T), we get $T = \sigma^*(T)$ (equality, not isomorphism). Since $f^*(G)$ comes from $\mathbf{P}^1$, σ acts as the identity, on the fiber of $f^*(G)$ over each ramification point of f. Since T is saturated in $f^*(G)$, the same is true for T. By descent theory there is a subbundle T' of G such that $f^*(T') = T$. We have $\mu_{\alpha/2}(T', B') = (\mu_\alpha(T, B))/2 \geq (\mu_\alpha(f^*(G), f^*(W)))/2 = \mu_{\alpha/2}(G, W)$, contradicting the stability of (G, W). □

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