International Journal of Pure and Applied Mathematics

Volume 20 No. 4 2005, 507-512

TWO-PHASE ABS METHOD FOR SOLVING OVER-DETERMINED LINEAR INEQUALITIES SYSTEM

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Abstract: A method, called the multi-stage ABS algorithm, for solving a system of linear inequalities is presented. This method is characterized by giving the explicit solution of linear inequalities system in finite steps, and it can determine the compatible of the system.

AMS Subject Classification: 65H10, 65F10, 65K05, 65Y05, 65Y10
Key Words: ABS algorithms, system of linear inequalities

1. Introduction

Consider the following system of linear inequalities

$$Ax \leq b, \tag{1}$$

where $A = (a_1, \cdots, a_m)^T \in \mathbb{R}^{m,n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $m > n$, and m, n are any positive. Suppose A is regular. Various methods have been designed for solving system (1), or for solving systems of equalities and inequalities via the ABS algorithms, for instance, a method due to Esmaeili, Mahdavi-Amiri, [3]. Esmaeili, Mahdavi-Amiri and Spedicato, proposed a method, called simply as EMS method, for solving system (1) under the assumptions that A is full rank in

Received: March 26, 2005

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row, i.e., the rank of A, $\mathrm{r}(A) = m$ and $m \leq n$, see [3] [4]. It can be proved, under these assumptions, that $Ax \leq d \Longleftrightarrow \{Ax = y,\, y \leq d\}$. An explicit general solution set is $x \in \{H_1^T W_m (A_m^T H_1^T W_m)^{-T} y + H_{m+1}^T H_1^{-T} x_1 + H_{m+1}^T q \,|\, y \leq d,\, q \in I\!R^n\}$, where $y \in I\!R^m$, H_1 is an arbitrary nonsingular matrix and x_1 is an arbitrary starting point. Shi [5] proposed a globally convergent ABS algorithm for generating a nonnegative solution to a linear system being of the form $\{Ax = b,\, x \geq 0, \quad A \in I\!R^{m,n}, b \in I\!R^m, x \in I\!R^n, m \leq n\}$ by carrying out a special iteration via the Huang algorithms so that iterative points asymptomatically satisfy the nonnegative conditions. A projection algorithm of the ABS form, equivalent to the simplex method via Bland's rules to a linear programming, for finding a feasible point of a general system of linear inequalities $\{Ax \geq b, A \in I\!R^{m,n}, b \in I\!R^m, x \in I\!R^n, m \leq n\}$, in a finite number of steps was constructed, and the compatibility for a system of linear inequalities with deficient rank one was investigated due to Zhang [6]. An algorithm for the least Euclidean norm solution of a linear system of inequalities via the Huang ABS algorithm and the Goldfarb-Idnani Strategy was proposed by Zhang [7]. A class of direct methods and ABS algorithms for solving linear inequalities by Zhao (see [8] [9]).

We will present a multi-stage ABS method in Section 3. We now recall the basic (unscaled) ABS algorithm for solving the following system of linear equations

$$Ax = b\,, \tag{2}$$

where $A = (a_1, \cdots, a_m)^T \in I\!R^{m,n}$, $b \in I\!R^m$, $x \in I\!R^n$, initiated by Abaffy, Broyden and Spedicato (see [1], [2]).

There are many special properties of the above recursions, see for instance, Abaffy and Spedicato [2]. Among them the following one is the most important for the derivation in this paper: the general solution to $Ax = b$ is formulated in the form

$$x = x_{m+1} + H_{m+1}^T q, \tag{3}$$

where $q \in I\!R^n$ is arbitrary, i. e., the linear variety containing all solutions consists of the vectors formulated by $H_{i+1} = H_i - H_i a_i w_i^T H_i / w_i^T H_i a_i$. For solving the system (1), we construct an equivalent system. We show that general solutions of the system can be expressed using a special solution and the matrix H_{m+1} generated by the ABS algorithm.

2. The ABS Method for Solving Inequalities System

Consider the linear inequalities system

$$\begin{cases} Ax + Ss = b, \\ s \geq 0, \end{cases} \tag{4}$$

where $S = [0, I_{n,n}; I_{l,l}, 0]$, s is the slack variable, $(x, s)^T = (x_1, x_2, \cdots, x_n, s_1, s_2, \cdots, s_m)^T$, and $l = m - n$. Denote the matrix (A, S) by $G = (g_1, g_2, \cdots, g_m)^T$, the vector (x, s) by $\widehat{x}$. Denote the matrix $(g_1, g_2, \cdots, g_i)^T$ by G_i

Theorem 1. *The system (4) is equivalent to system (1).*

When $s_i \geq 0$, $i = 1, 2, \cdots, m$, if x is the solution of system (4), it must be the solution of system (1). On the contrary, we have the same conclusion.

Theorem 2. *A is the coefficient matrix of system (1), G is the matrix (A, S). If A is regular, then G is also regular.*

Proof. Let $G^{(i)}$, $A^{(i)}$ denote the ith principal submatrices of G and A. $A^{(i)} = G^{(i)}$, $\quad i \in [1, n]$. When $i \in (n, m]$, $G^{(i)} = [A^{(n)} 0; \widetilde{A}, I_{i-n}]$, where $\widetilde{A} \in I\!R^{i-n,n}$, $\widetilde{A} = (a_{n+1}, \cdots, a_i)^T$. We have the conclusion that $\det(G^{(i)}) \neq 0$, $i \in (n, m]$. □

We divide the proceed of searching the solution of system (4) into two parts. In part one, we use the implicit LU algorithm to solve equations system $G\widehat{x} = b$. In part two, we find the slack variable s, which satisfies $s \geq 0$.

In part one, our aim is to find the solution of the equation $Ax + Ss = b$ in (4). Using the implicit LU algorithm to get the general solution of the equations system $G\widehat{x} = b$. The solution is $\widehat{x} = \widehat{x}^{*T} + H_{m+1}^T q, \quad q \in I\!R^{m+n}$, where $H_{m+1} \in I\!R^{m+n,m+n}$,

$$H_{m+1} = \begin{pmatrix} 0 & 0 \\ K_m & I_n \end{pmatrix}, \tag{5}$$

with $K_m = (K_1, K_2) \in I\!R^{n,m}$, $K_1 \in I\!R^{n,n}$, $K_2 \in I\!R^{n,m-n}$.

Lemma 2. *If the inverse matrix of C_1 exists, the inverse matrix of $A = [C_1, C_2; 0, I]$ is $A^{-1} = [C_1^{-1}, -C_1^{-1}C_2; 0, I]$.*

Theorem 3. *If the coefficient matrix G of the system (4) is regular, then the matrix K_m in (5) has the following structure $K_m = [B_1^{-1}, B_1^{-1}B_2]$, where the matrix B_1, B_2 is as following*

$$B_1 = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} a_{n+11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{n+1n} & \cdots & a_{mn} \end{pmatrix}$$

Proof. The matrix has the formulation as following

$$(K_1, K_2) = -E^T_{(m+n)-m} G_m (E^T_m G_m)^{-1} = [I_n, 0][B_1^{-1}, -B_1^{-1}B_2; 0, I].$$

Since $[B_1^{-1}, -B_1^{-1}B_2] \in R^{n,m+n}$, we have that $K_m = [B_1^{-1}, B_1^{-1}B_2]$. □

We have proved that x is the solution of system (1) if and only if $s \geq 0$, then we search the parameter q that satisfy the inequalities system $s \geq 0$. Since the structure of K_2 is very particular, we can get the range of q easily.

In part two, find the parameter q that satisfy the condition $s \geq 0$, that is

$$s^T = s^{*T} + \widehat{K}q \geq 0, \tag{6}$$

where $\widehat{K} = [0, K_2^T; 0, I_n]$, $K_2 = [B_1^{-1}, -B_1^{-1}B_2]$ by Theorem 3. Therefore (6) can be write as following

$$\begin{cases} \overline{s} + M\overline{q} \geq 0\,, \\ q_{m+i} \geq -s^*_{m+i}\,, \end{cases} \tag{7}$$

where $M = -B_2^T B_1^{-T}$. We get that the structure of K_2 has relation with $m-n$ and n. So one has the solution of inequalities system (7) in various situations.

$m-n$	q	i
$m-n>n$	$-s^*_{m-n+i} \leq q_{m+i} \leq s^*_{m-2n+i}$	$i = 1, 2, \cdots, n$
$m-n=n$	$-s^*_{n+i} \leq q_{m+i} \leq s^*_i$	$i = 1, 2, \cdots, n$
$m-n<n$	$-s^*_{m-n+i} \leq q_{m+i} \leq s^*_i$	$i \in [1, m-n]$
	$-s^*_{m-n+i} \leq q_{m+i}$	$i \in (m-n, n]$

Table 1: The range of parameter q to various $m-n$

Then we give the multi-stage ABS method to solve the system (4).

The Multi-Stage ABS Method

Let $H_1 = I$, $z_i = e_i$, $w_i = e_i / e_i^T H_i g_i$, where e_i is the i-th unit vector in $I\!R^{m+n}$.

(A) Initialization. Give an arbitrary vector $\widehat{x}_1 \in I\!R^{m+n}$ and H_1. Set $i = 1$ and iflag=0.

(B) Compute $\tau_i = \tau^T e_i = g_i^T \widehat{x}_i - b^T e_i$.

(D) Compute the search vector $p_i \in I\!R^{m+n}$ by $p_i = H_i^T z_i$.

(E) Update the approximation $\widehat{x}_i$ of a solution by $\widehat{x}_{i+1} = \widehat{x}_i - \alpha_i p_i$, where the stepsize α_i is computed by $\alpha_i = \tau_i / g_i^T p_i$. If $i = m$, stop; $\widehat{x}_{m+1}$ solves system $G\widehat{x} = b$. Goto (**H**).

(F) Update the (Abaffian) matrix H_i. Compute

$$H_{i+1} = H_i - H_i g_i w_i^T H_i / w_i^T H_i g_i .$$

(G) Increment the index i by one and goto (**B**).

(H) Solve equations system (6). If the solution set is empty, stop; the system (1) is incompatible.

(I) $x = x^* + K_1^T \widehat{q}$ is the solution of system (1), where $\widehat{q} \in I\!R^n$ satisfy (6).

Theorem 4. *The multi-stage ABS method can be implemented with no more than* $(m^3 - n^3)/6 + m^2 n + n^2 m + o(n^2)$ *multiplications.*

Proof. In the evaluation of H_{i+1}, no more than

$$\begin{cases} (m+n-i)(i+1)\,, & i \in [1,n]\,, \\ (m+n-i)(n+1)\,, & i \in (n,m]\,. \end{cases}$$

multiplications are required for $g_i = (g_{i1}, g_{i2}, \cdots, g_{in}, 0, \cdots, 0, 1, \cdots, 0)^T$ where 1 is the $(m+i)$-th element of g_i when $i \in [1,n]$, and 1 is the $(n+i)$-th element of g_i when $i \in (n,m]$. Multiplications are required for computing $H_i g_i$; no more than $(m+n-i)i$ multiplications are required for computing the nonzero elements of $H_i g_i s_i^T$. Sum all terms, the multiplications of the multi-stage ABS method are $(m^3 - n^3)/6 + m^2 n + n^2 m + o(n^2)$. □

3. Conclusion

We have proposed an approach for finding solutions of linear inequalities system. This approach is based on the implicit LU algorithm for solving system of linear equations and the special properties of it. By the multi-stage ABS method we can determine whether the solution exists or not. Moreover, if system (1) have solution, we can obtain the form of solutions.

Acknowledgements

The authors are supported by the Dr Research Foundation of Nationalities Univ., No. 20056202.

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