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STAR-PRODUCT ON NEWTON MAP FOR QUINTIC POLYNOMIALS

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Abstract: We introduce a tree structure for the iterates on Newton map for the family of quintic polynomials $f(x) = x^5 - c\ x + 1$. We identify a subset which we prove to be isomorphic to the family of unimodal maps. Finally, we give some properties about self-similarity and topological entropy for this product.

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1. Introduction and Motivation

Many mathematical problems can be reduced to compute the solutions of $f(x) = 0$, and Newton's method

$$x_{n+1} = N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ...$$

is the most common algorithm to solve this problem. The geometric interpre-

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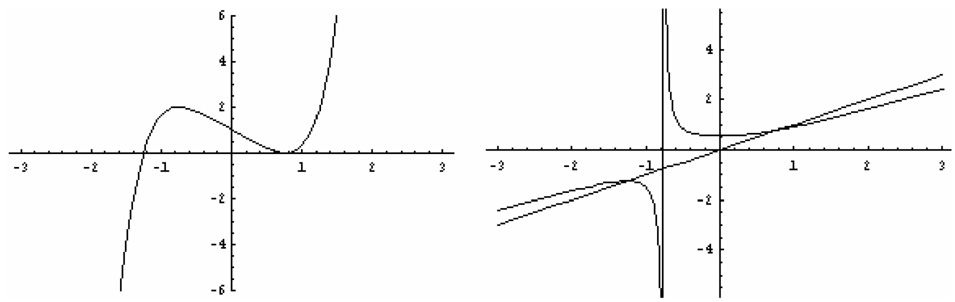

Figure 1: Graphs of $f_c(x) = x^5 - c\, x + 1$ and Newton map associated to $f_c(x)$, with $c = 5 \times 2^{-8/5}$

tation of the Newton's method is well known. In such a case x_{n+1} is the point where the tangent line $y - f(x_n) = f'(x_n)(x - x_n)$ of function $f(x)$ at point $(x_n, f(x_n))$ intersects the x-axis.

The fundamental property of Newton's method is that it transforms the problem of finding roots of $f(x)$ into the problem of finding attracting fixed points of Newton's map $N_f(x)$ as we can see in Figure 1.

As is well-known, the equation

$$x^5 + c_1 x^4 + c_2 x^3 + c_3 x^2 + c_4 x + c_5 = 0,$$

with arbitrary coefficients c_j, can be transformed to the *Bring-Jerrard* type

$$x^5 + a\, x + b = 0,$$

by a Tschirnhaus transformation [3, pp. 212-214].

Here we study the Newton map $N_{f_c} = x_n - f_c(x_n)/f_c'(x_n)$ with $f_c(x) = x^5 - c\, x + 1$, so in this case

$$N_{f_c}'(x) = \frac{f_c''(x) f_c(x)}{\left(f_c'(x)\right)^2} = \frac{20\ x^3 f_c(x)}{\left(f_c'(x)\right)^2}.$$

If $N_{f_c}'(x) = 0$ we have $x = 0$ or $f_c(x) = 0$.

When $c = 5 \times 2^{-8/5} = 1.64938...$ the relative minimum of N_{f_c} is 0 as we can see in Figure 1. We denote the parameter $c = 5 \times 2^{-8/5}$ by c_0. We use parameter values c between 0 and c_0.

As the roots of $f_c(x)$ are super-stable fixed points the only interesting critical point of N_{f_c} is 0 and we denote it by d_2, so for the study of iteration of N_{f_c} we will start at $x_0 = d_2$.

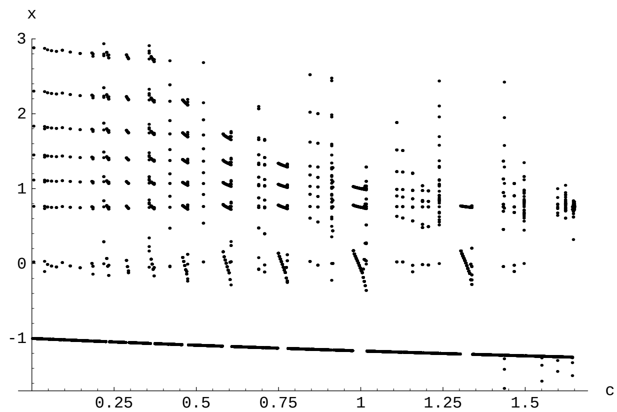

Figure 2: Bifurcation diagram for Newton map associated to $f_c(x) = x^5 - c\,x + 1$, with $0 < c < c_0 = 1.649...$

Let us now describe the numerical experiments which were performed in the c-parameter plane. To investigate this behavior further, we compute the bifurcation diagram in which the point $(c, N_{f_c}^n(0))$ is plotted, as we can see in Figure 2.

The outline of the paper is as follows. In Section 2, using standard symbolic dynamics, we introduce the admissibility rules of the sequences associated to Newton maps N_{f_c}. Then we study the structure of the set of admissible sequences. The techniques of symbolic dynamics are based on the notions of the kneading theory for one-dimensional multimodal maps, see Milnor and Thurston [7]. We construct a kneading sequences tree for the Newton map N_{f_c}.

In Section 3, we develop the star-product associated to Newton map of the quintic $f_c(x) = x^5 - c\,x + 1$ and we give some properties for this product.

2. Symbolic Dynamics

Kneading theory is an appropriate tool to classify topologically the dynamics of maps.

As we did before $N_{f_c}^n(x)$ is the Newton map for $f_c(x) = x^5 - c\,x + 1$ and we use $0 < c < c_0$. In this case $f_c(x)$ has only one root that we denote it by d_0. We have $f_c'(x) = 5x^4 - c$, so the polynomial $f_c(x) = x^5 - c\,x + 1$ has a relative maximum at $d_1 = -\sqrt[4]{c/5}$ and a relative minimum at $d_3 = \sqrt[4]{c/5}$.

We introduce the symbolic dynamics for the map N_{f_c} with $f_c(x) = x^5 - c\,x + 1$, where $0 < c < c_0$.

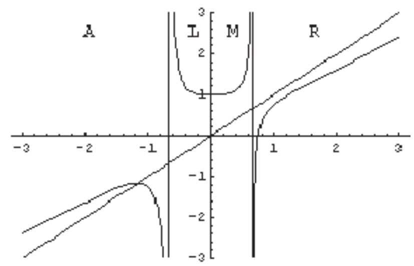

Figure 3: Subintervals and its associated symbols

We consider the alphabet $\mathcal{A} = \{A, L, C, M, R\}$, and the set $\Omega = \mathcal{A}^{\mathbb{N}_0}$ of symbolic sequences on the elements of $\mathcal{A}$. Now we take the application

$$i_c : \mathbb{R} \backslash \bigcup_{n \in \mathbb{N}_0} N_{f_c}^{-n}\left(\{d_1, d_3\}\right) \to \Omega$$

defined by

$$i_c(x)_m = \begin{cases} A & \text{if } N_{f_c}^m(x) < d_1 , \\ L & \text{if } d_1 < N_{f_c}^m(x) < d_2 , \\ C & \text{if } N_{f_c}^m(x) = d_2 , \\ M & \text{if } d_2 < N_{f_c}^m(x) < d_3 , \\ R & \text{if } N_{f_c}^m(x) > d_3 , \end{cases}$$

as we can see in Figure 3.

Remark 1. The interval corresponding to the symbol A contains two intervals of monotonicity, so following the usual techniques of symbolic dynamics it should correspond two symbols. But since $N_{f_c}^k(x) < d_1 \Longrightarrow N_{f_c}^m(x) < d_0$ $(\forall m > k)$ and $N_{f_c}^m(x) \underset{n}{\longrightarrow} d_0$, so we do not need to introduce another symbol.

If we now consider the shift operator $\sigma : \Omega \to \Omega$, $\sigma(X_0 X_1 X_2 ...) = X_1 X_2 X_3 ...$, we have the commutative diagram

$$\begin{array}{ccccc} & & N_{f_c} & & \\ & \Lambda & \longrightarrow & \Lambda & \\ i_c & \downarrow & & \downarrow & i_c \\ & \Omega & \longrightarrow & \Omega & \\ & & \sigma & & \end{array} ,$$

where

$$\Lambda = \mathbb{R} \backslash \bigcup_{n \in \mathbb{N}} N_{f_c}^{-n}\left(\{d_1, d_3\}\right).$$

We can introduce in Ω an order, induced lexicographically by the order in $\mathbb{R}$, with parity introduced by the subintervals, where the function is decreasing, $A < B < L < C < M < R$ when it is even and $-R < -M < C < -L < -B < -A$ when it is odd.

Definition 1. We say that $X \prec Y$ for $X, Y \in \Omega$, iff:

$$\exists_k : X_i = Y_i, \forall 0 \leq i < k \text{ and } (-1)^{n_L(X_1...X_{k-1})} X_k < (-1)^{n_L(X_1...X_{k-1})} Y_k \,,$$

where $n_L(X_1...X_{k-1})$ is equal to the number of times that the symbol L appears in $X_1...X_{k-1}$.

For example we have $MRRM... \prec MRRR...$ and $RLRA... \succ RLRR...$

Proposition 1. *Let $x, y \in \Lambda$. Then:*
i) $x < y \Longrightarrow i_c(x) \preceq i_c(y)$, *ii)* $i_c(x) \preceq i_c(y) \Longrightarrow x < y$.

Proof. We can adapt the proof in Milnor and Thurston [7]. □

We define the kneading sequence of the orbit of the critical point $x = d_2$ by

$$i : J \to \Omega, \qquad c \mapsto \sigma(i_c(d_2))\,,$$

with

$$J = \{c : c \in \,]0, c_0[\, \text{ and } c \text{ is such that } \bigcup_{n \in \mathbb{N}_0} N^n_{f_c}(d_2) \cap \{d_1, d_3\} = \emptyset\}.$$

Now we characterize the admissible sequences looking to the typical graph of $N^n_{f_c}$ (see Figure 3). We get the following matrix of admissibility T, where rows and columns are labeled by the elements of $\mathcal{A}$.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let Y be the symbolic sequence corresponding to the critical point orbit. Then, as the critical point d_2 is a local minimum, we get the following admissibility

$$\begin{cases} \sigma^i(Y)_1 = A \Rightarrow \sigma^{i+1}(Y) = A^\infty \,, \\ \sigma^i(Y)_1 = L \Rightarrow \sigma^{i+1}(Y) \geq Y \,, \\ \sigma^i(Y)_1 = M \Rightarrow \sigma^{i+1}(Y) \geq Y \,. \end{cases} \tag{2.1}$$

Let the set $\Omega^+ = \{Y \in \mathcal{A}^{\mathbb{N}} : Y \text{ verifies } T_{Y_i, Y_{i+1}} = 1 \text{ and } (2.1)\}$. We call Ω^+ the set of admissible sequences.

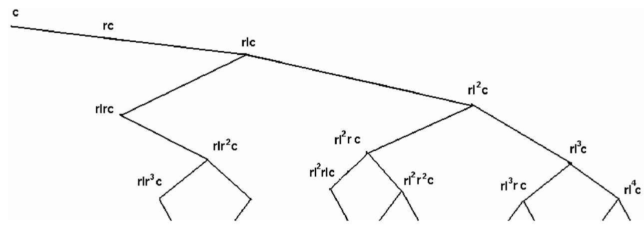

Figure 4: Tree T_{uni} of the admissible unimodal kneading sequences

Remark 2. To see the admissibility we must pay attention to the fact that the critical point d_2 is a local minimum and, in such case, if we have $\sigma^i(Y)_1 = L$ or $\sigma^i(Y)_1 = M$ then we must have $\sigma^i(Y) > Y$.

Example 1. According to the previous remark the sequence $(RLRC)^\infty$ is admissible – its occurrence can be seen in Figure 5 near $c = 1.3346...$ – while the sequences $(LMAC)^\infty$ and $(RMRC)^\infty$ are not admissible.

3. Star-Product

Derrida, A. Gervois and Y. Pomeau [2] introduced a $\star$-product between unimodal kneading sequences. This tool is suitable to formalize certain properties, in particular self-similarity as we do here, see also [9].

We denote by $\mathcal{K}_1$ the set of all admissible unimodal kneading sequences.

A tree structure T_{uni} can be associated to the couple $(\mathcal{K}_1, <)$ as follows. Each vertex of T_{uni} is a maximal sequence, that is, a sequence of the type $Q^{(q-1)}c = Q_1...Q_{q-1}c$ for some $q \in \mathbb{N}$. To each edge in the tree corresponds a new iteration. So, for each edge we can associate either a r or a ℓ, according to the place (right or left) where the iterated value of the critical point falls. In Figure 4 we can see the tree T_{uni} of the admissible unimodal kneading sequences with r–parity.

We introduce the star-product in Newton map for the family of quintic polynomials $f_c(x) = x^5 - cx + 1$.

Definition 2. We define the operation ($*$-product) $\star : \Omega^+ \times \mathcal{K}_1 \to \Omega^+$ on the following way: let $P = P_1...P_{p-1}C = P^{(p-1)}C \in \Omega^+$ and $Q = Q_1...Q_{q-1}c = Q^{(q-1)}c \in \mathcal{K}_1$ then

$$P \star Q = P^{(p-1)}C \star Q^{(q-1)}c = P^{(p-1)}\widetilde{Q}_1...P^{(p-1)}\widetilde{Q}_{q-1}P^{(p-1)}C\,,$$

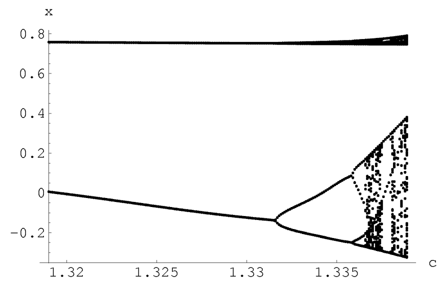

Figure 5: Bifurcation diagram with $c \in [1.32, 1.334]$, where we can see periods of $2^n k$, with $k, n \in \mathbb{N}$

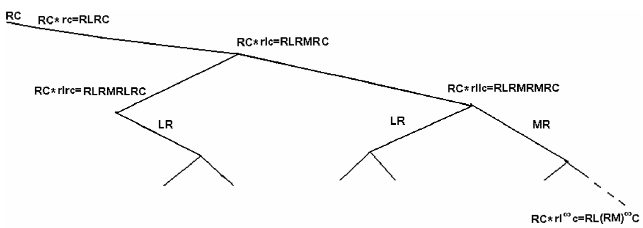

Figure 6: Tree for the $\star$-product $RC \star Q$, with $RC \in \Omega^+$ and $Q \in \mathcal{K}_1$

where

$$\widetilde{Q}_k = \begin{cases} L, & \text{if} \quad Q_k = r\,, \\ M, & \text{if} \quad Q_k = \ell\,, \end{cases}$$

when $n_L(P)$ is even and

$$\widetilde{Q}_k = \begin{cases} L, & \text{if} \quad Q_k = r\,, \\ M, & \text{if} \quad Q_k = \ell\,, \end{cases}$$

when $n_L(P)$ is odd, with $1 \leq k \leq q-1$.

Let us first see the example $RC \star Q$, with $RC \in \Omega^+$ and $Q \in \mathcal{K}_1$.

In Figure 6 we can see the tree of kneading sequences given by $RC \star Q$, with $RC \in \Omega^+$ and $Q \in \mathcal{K}_1$.

Remark 3. In the tree of Figure 6 the capital letters correspond to the sequences of Newton map for quintic polynomials of the form $f_c(x) = x^5 - cx + 1$, while the lower case letters correspond to the unimodal kneading sequences.

Corollary 1. *Let $RC \in \Omega^+$ and let T_{uni} be the tree of admissible unimodal kneading sequences, then we have the following:*

$$T_{RC} = RC \star T_{uni}.$$

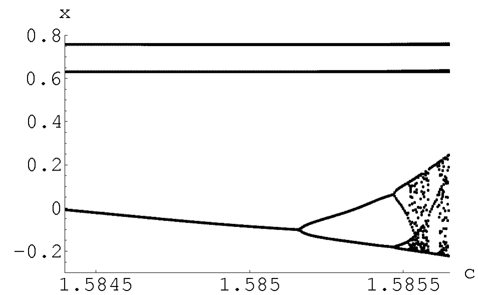

Figure 7: Bifurcation diagram with $c \in [1.584, 1.5855]$

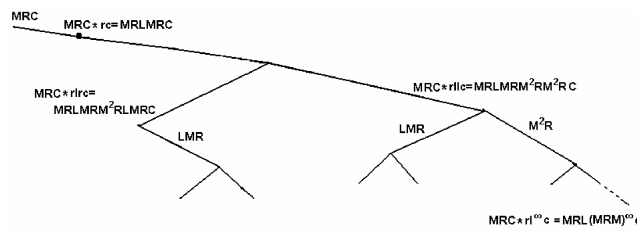

Figure 8: Tree for the $\star$-product $MRC \star Q$, with $MRC \in \Omega^+$ and $Q \in \mathcal{K}_1$

We study another example given by $MRC \star Q$, with $MRC \in \Omega^+$ and $Q \in \mathcal{K}_1$.

In Figure 8 we can see the tree of kneading sequences given by $MRC \star Q$, with $MRC \in \Omega^+$ and $Q \in \mathcal{K}_1$.

Again in the tree of Figure 8 the capital letters correspond to the sequences of Newton map, while the lower case letters correspond to the unimodal kneading sequences.

Corollary 2. *Let $MRC \in \Omega^+$ and let T_{uni} be the tree of admissible unimodal kneading sequences, then we have the following:*

$$T_{MRC} = MRC \star T_{uni}.$$

We can generalize the previous cases in the following theorem.

Theorem 1. *Let $P^{(p-1)}C$ be a periodic sequence with period p and let T_{uni} be the tree of the admissible unimodal kneading sequences, then the subtree $T_{P^{(p-1)}}C$ is constituted by all the admissible sequences located between its extreme branches $P^{(p-1)}C \star \ell^\infty$ and $P^{(p-1)}C \star r\ell^\infty$ and we have the following:*

$$T_{P^{(p-1)}C} = P^{(p-1)}C \star T_{uni}.$$

Proof. We can adapt the proof for the unimodal case in [5] but with one more symbol, respectively M. □

Theorem 2. *Between the extreme minimal $P^{(p-1)}C \star \ell^{\infty}$ and the extreme maximal $P^{(p-1)}C \star r\ell^{\infty}$ all the admissible sequences have the form $P^{(p-1)}C \star Q$, where $Q = Q^{(q-1)}c \in T_{uni}$.*

Proof. The proof is divided in two parts, each one analyzing the admissibility between the center branch $P^{(p-1)}C \star c$ and the extremes branches $P^{(p-1)}C \star \ell^{\infty}$ and $P^{(p-1)}C \star r\ell^{\infty}$.

The sequences between $P^{(p-1)}C \star \ell^{\infty}$ and $P^{(p-1)}C \star c$ are all no admissible sequences. We obtain this result by the action of the shift map, such none of the sequences is locally minimal.

In the second part there are several possible cases. We analyze one case and the other ones follow analogously. Suppose $P^{(p-1)}C$ has an even parity. If $S = P^{(p-1)}C \star Q$, with $Q \in T_{uni}$, then S is clearly admissible and $(P^{(p-1)}C)^{\infty} < P^{(p-1)}C \star Q < P^{(p-1)}C \star r\ell^{\infty}$, since the $\star$-product preserves both order and admissibility [1].

Let us now suppose S cannot be written in the previous form. In order that S be between the extreme branches of $T_{P^{(p-1)}C}$, its first $p-1$ symbols must be equal to the corresponding ones in $P^{(p-1)}$. In the following, analyzing exhaustively all the possible cases of $S = P_1 P_2 ... P_{p-1} S_p S_{p+1} ...$ We can show that we are always led to situations, where S is either non-admissible or not between the extremes of $T_{P^{(p-1)}C}$. □

In Misiurewicz and Szlenk [8] the topological entropy is determined by

$$h_{top}(N_{f_c}) = \log s(N_{f_c}),$$

where $s(N_{f_c}) = \lim_{k\to\infty} (L_k)^{1/k}$, L_k is the number of laps of $N_{f_c}^k$, i.e., the numbers of sub-intervals, where $N_{f_c}^k(x)$ is monotone (see also [7]).

When the orbit of the critical point d_2 of $N_{f_c}(x)$ is periodic we have a Markov partition which is determined by the itineraries of the critical point. Once we have a Markov partition, a subshift of finite type is determined by a transition matrix $\mathcal{M}$. Given a Markov partition $\mathcal{P} = \{I_j\}_{j=1}^m$, the transition matrix $\mathcal{M} = (a_{ij})$ of the type $(n \times n)$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad \operatorname{int}(N_{f_c}(I_i) \cap I_j) \neq \emptyset\,, \\ 0 & \text{if} \quad \operatorname{int}(N_{f_c}(I_i) \cap I_j) = \emptyset\,. \end{cases}$$

Like in [6] the topological entropy $h_{top}(N_{f_c})$ is obtained from the smallest real root t^* of $d_{\mathcal{M}}(t)$, where $d_{\mathcal{M}}(t) = \det(I - t\ \mathcal{M})$ is the characteristic polynomial of the transition matrix $\mathcal{M}$.

Also, we can compute $h_{top}(N_{f_c}) = \log 1/t^*$, where t^* is the minimal solution of $D(t^*) = 0$, see [6].

We also have the following property.

Theorem 3. *Let $P \in \Omega^+$ (with p length) and $Q \in \mathcal{K}_1$, then the $\star$-product preserves the topological entropy, that is:*

$$h_{top}(P \star Q) = h_{top}(P).$$

Proof. Applying kneading theory we can calculate the kneading determinant of the sequence $P \star Q$. The result is $D_{P\star Q}(t) = D_P(t) \times D_Q(t^p)$, where the smallest zero is the zero of $D_P(t)$ and we denote it by t_P. Then the topological entropy is $h_{top}(N_{P\star Q}) = \log \frac{1}{t_P} = h_{top}(N_P)$. □

The next step in our work is to study the types of bifurcations and the Hausdorff dimension of the set of badly initial points.

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