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A FACTORIZATION THEOREM RELATED TO COUSIN'S SECOND PROBLEM ON CERTAIN STEIN MANIFOLDS

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Abstract: Here we give a factorization theorem for any holomorphic fuction $f \neq 0$ on certain Stein manifolds as a product (finite or infinite) of a nowhere vanishing holomorphic function and "simpler functions", each of them vanishing exactly on one of the hypersurfaces whose union is the zero-locus of f.

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1. A Factorization Theorem for Holomorphic Functions on Certain Stein Manifolds

Our aim (only partially satisfied) is to give a factorization theorem for any holomorphic function $f \neq 0$ on a connected complex manifold X as a finite or infinite product of a nowhere vanishing holomorphic function (i.e. an invertible holomorphic function) and "simpler functions", each of them vanishing exactly on one of the hypersurfaces whose union is the zero-locus of f. If $X = \mathbb{C}$ or an open subset of $\mathbb{C}$, there is a classical factorization formula due to Weierstrass and Mittag-Leffler ([4], Theorems 15.9, 15.10 and 15.13). In higher dimension it is very natural to assume at least the existence of a holomorphic function vanishishing exactly on a prescribed hypersurface and with multiplity one. To

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assume the existence of many holomorphic functions we may, for instance, assume that X is Stein. In a Stein manifold the solution of the Cousin's second problem ([1], Theorem 5 at p. 143) makes reasonable to assume $H^2(X,\mathbb{Z}) = 0$. Here is our main result.

Theorem 1. *Let X be a connected complex Stein manifold such that $H^2(X,\mathbb{Z}) = 0$ and $f \in \mathcal{O}(X)\backslash\{0\}$. Assume the existence of an increasing sequence $\{K_n\}_{n\geq 1}$ of compact subsets of X such that $K_n = \widehat{K}_n$ (i.e. each K_n is holomorphically convex in X), $K_n \subset \overset{\circ}{K}_{n+1}$, $\cup_{n\geq 1}K_n = X$ and each K_n has a fundamental system of open Stein neighborhoods $\{U_{n,\alpha}\}$ with $H^1(U_{n,\alpha},\mathbb{Z}) = 0$. Let $Z = \cup_{i\in I} m_i Z_i$, $Z_i \neq Z_j$ for all $i \neq j$, be the Cartier divisor of all zeros of f with each Z_i reduced and irreducible and $m_i > 0$ the order of zero of f at a general point of Z_i. Hence I is countable (or empty) and the union $\cup_{i\in I} Z_i$ is locally finite. There is $f_i \in \mathcal{O}(X)$ such that $Z_i = \{f_i = 0\}$ and f_i has multiplicity one at a general point of Z_i. Fix any such set of functions $\{f_i\}_{i\in I}$. Then there are nowhere vanishing $h_i \in \mathcal{O}(X)$ such that $f = \prod_{i\in I} u_i$, where $u_i = f_i h_i$ and (if S is infinite) the infinite product on the right is uniformely absolutely convergent on every compact subset of X.*

The assumptions of Theorem 1 are satisfied if X is a convex open subset of $\mathbb{C}^n$. All our complex spaces are Hausdorff and with a countable basis.

For any reduce complex space Y we have an exact sequence ([1], p. 142) (the so-called exponential sequence):

$$0 \to \mathbb{Z} \to \mathcal{O}_Y \to {\mathcal{O}_Y}^* \to 1 \tag{1}$$

which allows one to compare Cartier divisors on Y, line bundles on Y and the cohomology groups $H^i(Y,\mathcal{O}_Y)$, $i = 1,2$, and $H^2(Y,\mathbb{Z})$. We summarize the parts we will use of this well-known story in the following remarks.

Remark 1. Let Y be a reduced complex space such that $H^1(Y,\mathcal{O}_Y) = H^2(Y,\mathcal{O}_Y) = 0$. Then the isomorphism classes of line bundles on Y are classified by $H^2(Y,\mathbb{Z})$. If $H^1(Y,\mathcal{O}_Y) = H^2(Y,\mathcal{O}_Y) = H^2(Y,\mathbb{Z}) = 0$, then for every effective Cartier divisor $Z \subset Y$, there is $f \in \mathcal{O}(Y) := H^0(Y,\mathcal{O}_Y)$ such that Z is the divisor of all zeros (counting the multiplities) of f (see [1], pp. 142–149).

Remark 2. Fix an integer $m \geq 1$. Let X be a reduced complex space such that $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$ and $m \geq 1$ a positive integer. For any effective Cartier divisor D of X there is $f \in \mathcal{O}(X)$ such that mD is the divisor of zeros of f .

Remark 3. Let Y be a reduced complex space such that $H^1(Y,\mathcal{O}_Y) = 0$. By the exponential sequence the following conditions are equivalent:

(i) for every nowhere vanishing $f \in \mathcal{O}(Y)$ there is $g \in \mathcal{O}(Y)$ such that $f = e^g$;

(ii) $H^1(Y, \mathbb{Z}) = 0$.

Proof of Theorem 1. By Theorem B of Cartan-Serre ([1],Ch IV, §1), we have $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Hence we may apply Remark 2. Since X is smooth, each Z_i is a Cartier divisor of X. Hence the existence of the holomorphic functions f_i follows from Remark 2. By Remark 2 there is $g \in \mathcal{O}(X)$ such that Z is the divisor of all zeros of g with the prescribed multiplicities $\{m_i\}_{i\in I}$. Since the result is obvious if I is finite by Remark 2, we may assume I infinite. Hence I is equipotent with $\mathbb{N}$. Fix an increasing sequence $\{K_n\}_{n\geq 1}$ of compact subsets of X as in the statement of Theorem 1. Set $I_0 = A_0 = \emptyset$. For all $n \geq 1$ set $I_n := \{i \in I : Z_i \cap K_n \neq \emptyset\}$ and $A_n := I_n \backslash I_{n-1}$. Taking if necessary a subsequence of the sequence $\{K_n\}_{n\geq 1}$ we may assume $A_n \neq \emptyset$ for all $n \geq 1$. Since the family $\{Z_i\}_{i\in I}$ is locally finite, each I_n is finite. Assume $n \geq 1$ and that we have defined all functions h_i (all of them nowhere vanishing) for all $i \in I_{n-1}$. Fix any $i_n \in A_n$ and set $h_i \equiv 1$ for all $i \in A_n \backslash \{i_n\}$. Since $f/(\prod_{\in I_n} f_i \times \prod_{i\in I_{n-1}} h_i)$ has neither zeros nor poles in K_n, the same is true on a suitable neighborhood $U_{n,\alpha}$. Since $H^1(U_{n,\alpha}, \mathbb{Z}) = 0$, Remark 3 implies the existence of $q_n \in \mathcal{O}(U_{n,\alpha})$ such that $e^{q_n} = f/(\prod_{\in I_n} f_i \times \prod_{i\in I_{n-1}} h_i)$. Since K_n is Runge in X, we may approximate q_n for the sup norm on K_n up to an arbitrary $\epsilon > 0$ by $w_n \in \mathcal{O}(X)$ ([2], Corollary 5.2.9). Set $h_{i_n} := e^{-w_n}$. For small ϵ we have $1 - 1/n^2 \leq |f(x)/\prod_{i\in I_n} f_i(x)h_i(x)| \leq 1 + 1/n^2$ for all $x \in K_n$. Thus $\lim_n(f/\prod_{i\in I_n} f_i h_i)$ defines a holomorphic function H on X. By construction $H(x) \neq 0$ for all $x \in X$. To get the statement of Theorem 1 multiply by H one of the functions h_i. □

Remark 4. The proof of Theorem 1 works verbatim (except for the quotation of [2], Corollary 5.2.9; in the singular case, use [3], Theorem 3.8) if instead to assume that X is smooth and connected we assume that X has only factorial singularities and that it is reduced and irreducible.

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