International Journal of Pure and Applied Mathematics
Volume 21 No. 2 2005, 167-174

STABILITY IN A CLASS OF MONOTONE NONLINEAR DIFFERENCE EQUATIONS

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Abstract: Conditions are given for the asymptotic stability of the unique equilibrium for difference equations of type

$$x_n = f\left(\sum_{i=1}^{m} [a_{n-i} g\left(x_{n-i}\right) + g_i\left(x_{n-i}\right)]\right), \quad n = 1, 2, 3, \dots ,$$

in which the maps f, g, g_i are all monotonic and the coefficients a_{n-i} are non-negative and m-periodic.

AMS Subject Classification: 39A11
Key Words: asymptotic stability, monotonic maps, periodic coefficients, rational difference equations

1. Introduction

Difference equations and recursions involving monotonic mappings have been studied in recent times with some vigor. Here we study the asymptotic stability of the equilibrium in the following type of difference equation

$$x_n = f\left(\sum_{i=1}^{m} [a_{n-i} g\left(x_{n-i}\right) + g_i\left(x_{n-i}\right)]\right), \quad n = 1, 2, 3, \dots , \tag{1}$$

where m is a fixed positive integer, and the variable coefficients are periodic and satisfy

Received: March 2, 2005

$$a_{mn+i} = a_i \geq 0,\ i = 1,\ldots,m,\ n = 1,2,3,\ldots,\ a = \sum_{i=1}^{m} a_i. \tag{2}$$

The functions f, g, g_i are all continuous on some interval (t_0, ∞) of real numbers $\mathbb{R}$ and monotonic (non-increasing or non-decreasing). It is assumed for non-triviality that all a_i and all g_i are not simultaneously zero. Define the function h as

$$h(t) = f\left(\sum_{i=0}^{m} g_i(t)\right), \quad g_0(t) = ag(t). \tag{3}$$

We assume the following about h in the sequel:

$$h(t) \text{ is decreasing for } t > t_0 \geq -\infty. \tag{4}$$

Special cases of equation (1) have appeared in the literature quite often. For instance, equations of type

$$x_n = \sum_{i=1}^{m} \frac{b_i}{x_{n-i}^{p_i}}$$

have been discussed in, e.g., [3, 4, 10-12]; also see Corollaries 2-4 below. Equation (1) generalizes the following equation for reentrant pulse propogation in a ring of excitable media (e.g., cardiac tissue)

$$x_n = \sum_{i=1}^{m} a_{n-i} C(x_{n-i}) - A(x_{n-m})$$

where C is decreasing and A is increasing on the positive real line (see [1, 2, 7, 14]). The preceding two equations are of type (1) with $f(t) = t$ in both cases. Also see [5], [6], [9, Section 2.5,2.6] and [13, Section 4.3D] for other variations of (1). In the next section we discuss general sufficient conditions for the existence and (non-local) stability of the equilibrium (1). These results are in particular applicable to the preceding two equations and many more that are similarly monotonic.

2. The Main Results

Lemma 1. *Under conditions (2) and (4) and the following additional property:*

$$h(r) > r \quad \textit{for some } r > t_0 \geq -\infty. \tag{5}$$

Equation (1) has a unique fixed point $\bar{x} > r$.

Proof. A fixed point of (1) is a solution of the equation $h(t) = t$. Define $H(t) = h(t) - t$ so that the fixed points of (1) is a zero of H. Note that $H(r) > 0$ by (5) and

$$H(h(r)) = h(h(r)) - h(r) < 0$$

because h is decreasing. Since H is continuous and decreasing it follows that it has a unique zero $\bar{x} \in (r, h(r))$. □

Lemma 2. *Under conditions (2), (4), (5) and the following inequality:*

$$h^2(s) = h(h(s)) \geq s \quad \text{for some } s \in (r, \bar{x}). \tag{6}$$

the interval $(s, h(s))$ is invariant for (1) and $\bar{x} \in (s, h(s))$.

Proof. Note that $h(s) > h(\bar{x}) = \bar{x} > s$ so that $\bar{x} \in (s, h(s))$. Let $x_0, x_{-1}, \ldots, x_{-m+1} \in (s, h(s))$. Then

$$x_1 = f\left(\sum_{i=1}^{m} [a_{1-i} g(x_{1-i}) + g_i\left(x_{1-i}\right)]\right) \leq f\left(\sum_{i=0}^{m} g_i\left(s\right)\right) = h(s)$$

and also

$$x_1 = f\left(\sum_{i=1}^{m} [a_{1-i} g(x_{1-i}) + g_i\left(x_{1-i}\right)]\right) \geq f\left(\sum_{i=1}^{m} g_i\left(h(s)\right)\right) = h^2(s) \geq s.$$

Thus $x_1 \in (s, h(s))$. Now inductively assume that $x_1, \ldots, x_k \in (s, h(s))$ for some $k \geq 1$. Then an argument similar to the one above for x_1 shows that $x_{k+1} \in (s, h(s))$. Hence $(s, h(s))$ is invariant for equation (1). □

Theorem. *Assume that (2), (4), (5) hold and that there is $s \in (r, \bar{x})$ such that*

$$h^2(t) = h(h(t)) > t \quad \text{for all } t \in (s, \bar{x}). \tag{7}$$

Then the fixed point $\bar{x}$ of Equation (1) is stable and attracts every point in the interval $(s, h(s))$.

Proof. First we establish the attracting nature of $\bar{x}$. Let $x_0, \ldots, x_{-m+1}$ be in the interval $(s, f(s))$, and define

$$\mu_1 = \min\{\bar{x}, x_0, \ldots, x_{-m+1}\}, \quad \mu_2 = \max\{\bar{x}, x_0, \ldots, x_{-m+1}\}.$$

Since h is continuous, we have $h(t) \to h(s)$ as $t \to s$. Thus we can find $q \in (s, \mu_1)$ sufficiently close to s that $h(q) \in (\mu_2, h(s))$. Next, observe that since $x_0, \ldots, x_{-m+1} > q$,

$$x_1 = f\left(\sum_{i=1}^{m} [a_{1-i} g(x_{1-i}) + g_i\left(x_{1-i}\right)]\right) \leq f\left(\sum_{i=0}^{m} g_i\left(q\right)\right) = h(q)\,.$$

Similarly, $x_0, \dots, x_{-m+1} < h(q)$ implies

$$x_1 = f\left(\sum_{i=1}^{m}[a_{1-i}g(x_{1-i}) + g_i\left(x_{1-i}\right)]\right) > f\left(\sum_{i=0}^{m} g_i\left(h(q)\right)\right) = h^2(q) > q.$$

Therefore, $x_1 \in (h^2(q), h(q)) \subset (q, h(q))$. Repeating a similar calculation for $x_2, \dots, x_m$ we conclude that

$$x_k \in (h^2(q), h(q)) \subset (q, h(q)), \quad k = 1, \dots, m. \tag{8}$$

Next, we move on to the next cycle and look at x_{m+1}. Since by (8) $x_1, \dots, x_m > h^2(q)$,

$$x_{m+1} = f\left(\sum_{i=1}^{m}[a_{m-i}g(x_{m-i}) + g_i\left(x_{m-i}\right)]\right) \leq f\left(\sum_{i=0}^{m} g_i\left(h^2(q)\right)\right) = h^3(q);$$

further, $x_1, \dots, x_m < h(q)$ gives

$$x_{m+1} = f\left(\sum_{i=1}^{m}[a_{m-i}g(x_{m-i}) + g_i\left(x_{m-i}\right)]\right) > f\left(\sum_{i=0}^{m} g_i\left(h(q)\right)\right) = h^2(q).$$

Since $h^3(q) < h(q)$, this argument can be repeated for $x_{m+2}, \dots, x_{2m}$ to yield

$$x_k \in (h^2(q), h^3(q)) \subset (h^2(q), h(q)), \quad k = m+1, \dots, 2m.$$

Continuing this argument inductively, we conclude that

$$\begin{aligned} x_k &\in (h^{2n}(q), h^{2n-1}(q)), \quad k = m(2n-2)+1, \dots, m(2n-1) \\ x_k &\in (h^{2n}(q), h^{2n+1}(q)), \quad k = m(2n-1)+1, \dots, 2mn. \end{aligned} \tag{9}$$

From these relations and the following claim it is easy to see that x_k converges to x^* as $k \to \infty$. □

Claim. *For every* $x_0 \in (s, \bar{x})$

$$s < x_0 < h^2(x_0) < \cdots < \bar{x} < \cdots < h^3(x_0) < h(x_0) < h(s) \tag{10}$$

and

$$\lim_{n\to\infty} h^{2k}(x_0) = \lim_{n\to\infty} h^{2k+1}(x_0) = \bar{x}. \tag{11}$$

Since g is decreasing, if $x_0 \in (s, \bar{x})$ then $h(x_0) > h(\bar{x}) = \bar{x}$ and $h(x_0) < h(s)$. Thus

$$\bar{x} < h(x_0) < h(s). \tag{12}$$

Applying g to (12) in the above fashion gives

$$h^2(s) < h^2(x_0) < \bar{x}.$$

Now (10) follows by simple induction. Statements (11) follow from (10) because h has no fixed points in $(s, h(s))$ other than $\bar{x}$ to which the odd and even iterates of h can converge. The claim is proved.

It remains to show that $\bar{x}$ is stable (dynamically in the sense of Liapunov). Let $\varepsilon > 0$ and use the continuity of h to pick $\delta \in (0, \varepsilon)$ small enough that $h(\bar{x} - \delta) < \bar{x} + \varepsilon$. If $x_0, \ldots, x_{-m+1} \in (\bar{x} - \delta, \bar{x} + \delta)$ then it follows from (10) and (9) that

$$x_k \in (\bar{x} - \delta, h(\bar{x} - \delta)) \subset (\bar{x} - \varepsilon, \bar{x} + \varepsilon)), \quad k \geq 1.$$

Hence $\bar{x}$ is stable.

Inequality (7) is necessary and sufficient for the asymptotic stability of the fixed point of h on the invariant interval $(s, h(s))$ (see [13, Theorem 2.1.2]). Further, under common differentiability hypotheses, the inequality $|h'(\bar{x})| < 1$ implies local asymptotic stability. Therefore, we have the following immediate corollary.

Corollary 1. *If $\bar{x}$ is an asymptotically stable fixed point of h then $\bar{x}$ is an asymptotically stable fixed point of (1).*

In particular, under suitable differentiability and invertibility hypotheses, $\bar{x}$ is an asymptotically stable fixed point of (1) if

$$\left| f'\left(\sum_{i=0}^{m} g_i(\bar{x})\right) \sum_{i=0}^{m} g_i'(\bar{x}) \right| < 1, \text{ or } \left| \sum_{i=0}^{m} g_i'(\bar{x}) \right| < \left| \frac{d}{dx} f^{-1}(\bar{x}) \right|.$$

The next three corollaries give a few applications of the above Theorem to difference equations of rational type. Equations similar to those encountered here have been studied in the literature using a variety of methods. For example, to study a special case of the equation in Corollary 2 below, [11] applies the method of “full limiting sequences” in [8]. Other methods involve semicycles or other arguements [3, 4, 10, 12]. Corollaries 2-4 below use the new monotonic method discussed above and treat the periodic coefficients case apparently for the first time.

Corollary 2. *Assume that* $p, q \in \mathbb{R}$, $b_i \geq 0$ *with* $\sum_{i=1}^{m} b_i = b$ *and the sequence* $\{a_n\}$ *satisfies (2). If* $a + b > 0$ *and* $0 < pq < 1$ *then the rational difference equation*

$$x_n = \left[\sum_{i=1}^{m} \frac{a_{n-i} + b_i}{x_{n-i}^p} \right]^q$$

has a unique positive fixed point $\bar{x}$ *that is stable and attracts every positive solution.*

Proof. Here the function h is given by

$$h(t) = \frac{c}{t^{pq}}, \quad c = (a+b)^q > 0, \; t > 0.$$

If $pq > 0$ then this mapping has a unique positive fixed point

$$\bar{x} = c^{1/(1+pq)} > 0.$$

Now, $h^2(t) = c^{1-pq} t^{p^2q^2}$ so that $h^2(t) > t$ if and only if

$$c^{1-pq} > t^{1-p^2q^2} = t^{(1+pq)(1-pq)}.$$

If $pq < 1$ then the above inequality reduces to $t^{1+pq} < c$ or equivalently, $0 < t < \bar{x}$ as required. □

Corollary 3. *Assume that* $\alpha > 0$, $b_i \geq 0$ *with* $\sum_{i=1}^{m} b_i = b$ *and the sequence* $\{a_n\}$ *satisfies (2). If* $a + b > 0$, *then the rational difference equation*

$$x_n = \frac{1}{\alpha + \sum_{i=1}^{m} (a_{n-i} + b_i) x_{n-i}} \tag{13}$$

has a unique positive fixed point $\bar{x}$ *that is stable and attracts every positive solution.*

Proof. In this case,

$$h(t) = \frac{1}{\alpha + ct}, \quad c = a + b > 0, \; t \geq 0$$

has a unique positive fixed point

$$\bar{x} = \frac{1}{2c} \left(-\alpha + \sqrt{\alpha^2 + 4c} \right).$$

It is easy to see that

$$h^2(t) - t = \frac{\alpha(1 - \alpha t - ct^2)}{c + \alpha^2 + c\alpha t} > 0 \quad \text{if} \quad 0 \leq t < \bar{x}$$

so $\bar{x}$ is stable and attracts every point of the invariant interval $(0, 1/\alpha)$. Since $x_n < 1/\alpha$ in (13) for $n \geq 1$ and positive initial values, the proof is complete. □

Corollary 4. *Assume that $\alpha > 0$, $b_i \geq 0$ with $\sum_{i=1}^{m} b_i = b$ and the sequence $\{a_n\}$ satisfies (2). If $a + b > 0$, then the rational difference equation*

$$x_n = \alpha + \sum_{i=1}^{m} \frac{a_{n-i} + b_i}{x_{n-i}}$$

has a unique positive fixed point $\bar{x}$ that is stable and attracts every positive solution.

Proof. Here the function h is given by

$$h(t) = \alpha + \frac{c}{t}, \quad c = a + b > 0,\ t > 0.$$

This mapping has a positive fixed point

$$\bar{x} = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4c}\right)$$

and since $\bar{x}$ is the only positive zero of the quadratic polynomial $c + \alpha t - t^2$, it is easy to see that

$$h^2(t) - t = \frac{\alpha(c + \alpha t - t^2)}{c + \alpha t} > 0 \quad \text{if} \quad 0 \leq t < \bar{x}.$$

Hence by the above theorem, $\bar{x}$ is stable and attracts every point of the invariant interval $(0, \infty)$. □

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