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ASYMPTOTIC BEHAVIOR OF THE ENERGY TO A THERMO-VISCOELASTIC MINDLIN-TIMOSHENKO PLATE WITH MEMORY

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Abstract: A thermo-viscoelastic model of a Mindlin-Timoshenko plate is considered. Uniform energy estimates are then given and the existence of an absorbing set for the solution of the problem is found.

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1. Introduction

In this paper we deal with a homogeneous, thermally and elastically isotropic Mindlin-Timoshenko plate subject to thermal deformations and hereditary heat conduction law. We assume that the plate is of uniform thickness $d > 0$, and, in equilibrium, it occupies a fixed bounded domain $\mathcal{D} \subset \mathbb{R}^3$ placed in a reference frame $\boldsymbol{x} = (x_1, x_2, x_3)$. The plate has a middle surface midway between its faces in a region $\Omega \subset \mathbb{R}^2$ of the plane $x_3 = 0$, with boundary $\Gamma = \partial\Omega$. Denote by $u(x_1, x_2; t)$ the *bending component* of the displacement vector of the point which, when the plate is in equilibrium, has coordinates $(x_1, x_2, 0)$ at time

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$t \geq 0$. $\psi(x_1, x_2; t)$ and $\phi(x_1, x_2; t)$ represent the *angles of rotation* of the cross section $x_1 =$ const., $x_2 =$ const. containing the filament which, when the plate is equilibrium, is perpendicular to the middle surface at the point $(x_1, x_2, 0)$ at time $t \geq 0$. Also, indicate by $\theta(x_1, x_2; t)$ the *normal thermal gradient* at the point $(x_1, x_2, 0)$ at time $t \geq 0$ at equilibrium. Set $Q = \Omega \times \mathbb{R}^+$ and $\Sigma = \Gamma \times \mathbb{R}^+$, and put $\boldsymbol{v} = -\begin{bmatrix} \psi \\ \phi \end{bmatrix}$. Then, the evolution of $(\boldsymbol{v}, u, \theta)$ can be written as (see [12])

$$
\begin{aligned}
&\partial_{tt}\boldsymbol{v}(t) - N(0)\mathcal{A}\boldsymbol{v}(t) + H(0)\left[\nabla u(t) + \boldsymbol{v}(t)\right] \\
&\qquad\qquad\qquad - \int_0^\infty N'(s)\mathcal{A}\boldsymbol{v}(t-s)ds \\
&+ \int_0^\infty H'(s)\left[\nabla u(t-s) + \boldsymbol{v}(t-s)\right]ds + \nabla\theta(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \\
&\partial_{tt}u(t) - H(0)\nabla\cdot(\boldsymbol{v}(t) + \nabla u(t)) \\
&\qquad - \int_0^\infty H'(s)\nabla\cdot\left[\nabla u(t-s) + \boldsymbol{v}(t-s)\right]ds = f_3(t), \\
&\partial_t\theta(t) - \int_0^\infty k(s)\Delta\theta(t-s)ds + \int_0^\infty g'(s)\theta(t-s)ds \\
&\qquad\qquad + g(0)\theta(t) + \nabla\cdot\partial_t v(t) = f_4(t),
\end{aligned}
\tag{1.1}
$$

in Q, where $N(0) > 0$, $H(0) > 0$ and $g(0) > 0$ are the *shear modulus of elasticity*, the *bending moment distribution* and the *coefficient depending on thermal conductivity*, respectively. N' and H' are the memory kernels related to viscoelastic effets, while k and g' are the memory kernels accounting for thermal effects. Functions $f_i(t)$, $i = 1, 2, 3, 4$, denote forces acting on the plate. The spacial operator $\mathcal{A}$ is denoted by

$$
\mathcal{A} = \begin{bmatrix} \dfrac{\partial^2}{\partial x_1^2} + \dfrac{1-\nu_0}{2}\dfrac{\partial^2}{\partial x_2^2} & \dfrac{1+\nu_0}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} \\ \dfrac{1+\nu_0}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} & \dfrac{1-\nu_0}{2}\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} \end{bmatrix}, \tag{1.2}
$$

where the *viscoelastic Poisson's ratio* ν_0 is assumed to be constant. Furthermore, we denote by $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$ and $\partial_t = \partial/\partial t$.

Boundary conditions we consider here are given by

$$
\boldsymbol{v}(t) = \mathbf{0}, \quad u(t) = 0, \quad \theta(t) = 0, \quad \text{on } \Sigma, \tag{1.3}
$$

whereas initial condition are

$$\begin{aligned} &\boldsymbol{v}(-t)=\boldsymbol{v}_0(t), \quad u(-t)=u_0(t), \quad \theta(-t)=\theta_0(t), \quad \text{in } \Omega\times[0,\infty),\\ &\boldsymbol{v}_t(0)=\boldsymbol{v}_1, \qquad u_t(0)=u_1, \qquad\qquad\qquad\qquad \text{in } \Omega, \end{aligned} \tag{1.4}$$

for some given functions $\boldsymbol{v}_0(t):\Omega\times[0,\infty)\to\mathbb{R}^2$, $u_0(t):\Omega\times[0,\infty)\to\mathbb{R}$, $\theta_0(t):\Omega\times[0,\infty)\to\mathbb{R}$, $\boldsymbol{v}_1:\Omega\to\mathbb{R}^2$, $u_1:\Omega\to\mathbb{R}$.

Our main interest concerns the longtime behavior of the energy of the system above, and in particular, whether the dissipation given by the memory effect in system (1.1) is strong enough to stabilize the whole system. As recalled in other papers [1, 7] the introduction of the dissipative terms into memory kernels do not produce any rate of decay if the relaxation functions do not decay uniformly. Then, we assume that the memory kernels exponentially decay in time. By uniform energy estimates, when external sources act on the plate, we show the existence of an absorbing set [2, 18, 19, 25] for the solutions to model (1.1)-(1.17); whereas without external sources, the energy decays exponentially, provided the memory kernels go to zero exponentially.

We also remark that the longtime behavior of the plate model is investigated by a direct approach of energy methods, using suitably sophisticated estimates for multipliers.

In order to carry out this program, it is convenient to translate (1.1)-(1.4) into a (linear) differential system on a suitable phase-space. Recalling the approach of Dafermos [3], we consider the past histories $\eta=\eta^t(s):\Omega\times\mathbb{R}^+\times[0,\infty)\to\mathbb{R}$, $\omega=\omega^t(s):\Omega\times\mathbb{R}^+\times[0,\infty)\to\mathbb{R}$, $\boldsymbol{\zeta}=\boldsymbol{\zeta}^t(s):\Omega\times\mathbb{R}^+\times[0,\infty)\to\mathbb{R}^2$, defined as

$$\eta^t(s)=\int_0^s\theta(t-\tau)d\tau, \tag{1.5}$$

$$\omega^t(s)=u(t)-u(t-s), \tag{1.6}$$

$$\boldsymbol{\zeta}^t(s)=\boldsymbol{v}(t)-\boldsymbol{v}(t-s). \tag{1.7}$$

For every $s\in\mathbb{R}^+$, we set

$$\begin{aligned} &-N'(s)=\nu(s), \qquad -H'(s)=\mu(s),\\ &-k'(s)=\kappa(s), \qquad -g''(s)=\iota_0\gamma(s), \end{aligned}$$

according to the following choices of g:

$$\iota_0=1 \qquad \text{if } g \text{ is bounded, nondecreasing and concave,} \tag{1.8}$$

$$\iota_0 = -1 \qquad \text{if } g \text{ is summable, non increasing and convex.} \tag{1.9}$$

Both hypothesis are thermodynamically consistent (see [10] and references therein). Set $N_\infty = \lim_{s\to+\infty} N(s)$, $H_\infty = \lim_{s\to+\infty} H(s)$, and assuming $\lim_{s\to+\infty} k(s) = 0$, $\lim_{s\to+\infty} g'(s) = 0$, a formal integration by parts in integral terms of (1.1) leads to

$$\int_0^\infty k(s)\Delta\theta(t-s)ds = \int_0^\infty k(s)\Delta\partial_s\eta^t(s)ds = -\int_0^\infty k'(s)\Delta\eta^t(s)ds,$$

$$\int_0^\infty g'(s)\theta(t-s)ds = -\int_0^\infty g''(s)\eta^t(s)ds,$$

$$\int_0^\infty N'(s)\mathcal{A}\mathbf{v}(t-s)ds = [N_\infty - N(0)]\mathcal{A}\mathbf{v}(t) - \int_0^\infty N'(s)\mathcal{A}\boldsymbol{\zeta}^t(s)ds,$$

$$\begin{aligned}&\int_0^\infty H'(s)[\nabla u(t-s) + \mathbf{v}(t-s)]ds \\ &\quad = [H_\infty - H(0)][\nabla u(t) + \mathbf{v}(t)] - \int_0^\infty H'(s)[\nabla\omega^t(s) + \boldsymbol{\zeta}^t(s)]ds.\end{aligned}$$

Without loss of generality, we set $N_\infty = H_\infty = 1$. Introducing $\mathcal{Q} = Q \times \mathbb{R}^+$, we end up with the following system

$$\begin{aligned}&\partial_{tt}\boldsymbol{v}(t) - \mathcal{A}\boldsymbol{v}(t) + \nabla u(t) + \boldsymbol{v}(t) + \nabla\theta(t) \\ &\quad - \int_0^\infty \nu(s)\mathcal{A}\boldsymbol{\zeta}^t(s)ds \\ &\quad + \int_0^\infty \mu(s)[\nabla\omega^t(s) + \boldsymbol{\zeta}^t(s)]ds = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \qquad \text{in } Q,\end{aligned} \tag{1.10}$$

$$\begin{aligned}&\partial_{tt}u(t) - \nabla\cdot[\boldsymbol{v}(t) + \nabla u(t)] \\ &\quad - \int_0^\infty \mu(s)\nabla\cdot[\nabla\omega^t(s) + \boldsymbol{\zeta}^t(s)]ds = f_3(t) \qquad \text{in } Q,\end{aligned} \tag{1.11}$$

$$\begin{aligned}&\partial_t\theta(t) - \int_0^\infty \kappa(s)\Delta\eta^t(s)ds + \iota_0\int_0^\infty \gamma(s)\eta^t(s)ds \\ &\quad + \theta(t) + \nabla\cdot\partial_t\boldsymbol{v}(t) = f_4(t) \qquad \text{in } Q,\end{aligned} \tag{1.12}$$

$$\partial_t\boldsymbol{\zeta}^t(s) + \partial_s\boldsymbol{\zeta}^t(s) = \partial_t\boldsymbol{v}(t) \qquad \text{in } \mathcal{Q}, \tag{1.13}$$

$$\partial_t\omega^t(s) + \partial_s\omega^t(s) = \partial_t u(t) \qquad \text{in } \mathcal{Q}, \tag{1.14}$$

$$\partial_t \eta^t(s) + \partial_s \eta^t(s) = \theta(t) \qquad \text{in } \mathcal{Q}. \tag{1.15}$$

Initial and boundary conditions become

$$\begin{array}{lll}
\boldsymbol{v}(0) = \boldsymbol{v}_0, & \boldsymbol{v}_t(0) = \boldsymbol{v}_1, & \text{in } \Omega,\\
u(0) = u_0, & u_t(0) = u_1, & \text{in } \Omega,\\
\theta(0) = \theta_0, & & \text{in } \Omega,\\
\boldsymbol{\zeta}^0(s) = \boldsymbol{\zeta}_0(s), \quad \omega^0(s) = \omega_0(s), & \eta^0(s) = \eta_0(s), & \text{in } \Omega \times \mathbb{R}^+,
\end{array} \tag{1.16}$$

having set $\boldsymbol{v}_0 = \boldsymbol{v}_0(0)$, $u_0 = u_0(0)$, $\boldsymbol{\zeta}_0(s) = \boldsymbol{v}_0(0) - \boldsymbol{v}_0(s)$, $\omega_0(s) = u_0(0) - u_0(s)$, $\eta_0(s) = \displaystyle\int_0^s \theta_0(\tau)\, d\tau$, and

$$\begin{array}{ll}
\boldsymbol{v}(t) = \mathbf{0},\ u(t) = 0,\ \theta(t) = 0, & \text{on } \Gamma,\ t \geq 0,\\
\boldsymbol{\zeta}^t(s) = \mathbf{0},\ \omega^t(s) = 0,\ \eta^t(s) = 0, & \text{on } \Gamma,\ s > 0,\ t \geq 0,\\
\boldsymbol{\zeta}^t(0) = \lim_{s\to 0} \boldsymbol{\zeta}^t(s) = \mathbf{0}, & \text{in } \Omega,\ t \geq 0,\\
\omega^t(0) = \lim_{s\to 0} \omega^t(s) = 0, & \text{in } \Omega,\ t \geq 0,\\
\eta^t(0) = \lim_{s\to 0} \eta^t(s) = 0, & \text{in } \Omega,\ t \geq 0.
\end{array} \tag{1.17}$$

We briefly sketch the plan of the paper. In Section 2 we recall the constitutive equations considered for our model and resume some previous results presented in literature and related to our problem. Section 3 contains conditions on memory kernelys and external sources of the plate, functional setting and notation used in the paper. In Section 4 well-posedness of the Cauchy-Dirichlet problem is recalled. The main results are stated in Section 5 and proved in latter sections via uniform energy estimates.

2. Thermo-Visco-Elastic Mindlin-Timoshenko Plate Model and Literature

The material composing the plate is homogeneous and (elastically and thermally) *isotropic*, so that its stress-strain law is given by

$$\mathbf{T}(\boldsymbol{x},t) = \mathbb{L}(0)\boldsymbol{\varepsilon}(\boldsymbol{x},t) + \int_0^\infty \mathbb{L}'(\tau)\boldsymbol{\varepsilon}(\boldsymbol{x},t-\tau)d\tau - \mathbb{L}(0)\alpha_0\vartheta(\boldsymbol{x},t)\mathbf{I}, \tag{2.1}$$

where the *elastic strain* $\boldsymbol{\varepsilon}$, the *stress* $\mathbf{T}$ are second-order tensors, $\mathbf{I}$ is the second-order identity tensor. Furthermore, at any fixed $\tau \in \mathbb{R}$, $\mathbb{L}(\tau)$ is an isotropic fourth-order tensor which vanishes for $\tau < 0$ and involves two independent *relaxations functions* l and m, such that

$$\mathbb{L}(\tau) = l(\tau)\,\mathbf{I}\otimes\mathbf{I} + 2\,m(\tau)\,\mathbb{I}\,, \qquad \text{for } \tau \geq 0\,. \tag{2.2}$$

The last term in (2.1) represents the *thermal strain* and the positive constant α_0 is called the *coefficient of thermal expansion.* Moreover, $\vartheta := \Theta - \Theta_0$ denotes the *variation* of the *absolute temperature* Θ with respect to a *reference value* Θ_0. In order to account for small temperature variations inside the plate, we assume that ϑ obeys the approximate relation

$$\vartheta(\boldsymbol{x},t) = \vartheta(x_1,x_2,x_3;t) = \bar{\theta}(x_1,x_2;t) + x_3\,\theta(x_1,x_2;t), \tag{2.3}$$

where $\bar{\theta}$ and θ denote respectively the *temperature of the middle surface* and the *normal thermal gradient*.

Let $\boldsymbol{q}\colon \Omega\times\mathbb{R} \to \mathbb{R}^3$ be the *mean heat flux vector* in the plate. Gurtin-Pipkin heat flux for a thermally isotropic body is

$$\mathbf{q}(\boldsymbol{x},t) = -\int_0^\infty K(\tau)\ \nabla\vartheta(\boldsymbol{x},t-\tau)\ d\tau, \tag{2.4}$$

where $K\colon \mathbb{R}^+ \to \mathbb{R}$ is the *heat flux memory kernel.*

The usual energy balance equation is replaced by

$$\rho_0\,h(\boldsymbol{x},t) = -\nabla\cdot\boldsymbol{q}(\boldsymbol{x},t) + \rho_0\,r(\boldsymbol{x},t)\,, \tag{2.5}$$

where h is the *thermal power*, which denotes the rate of heat absorption per unit of volume, $\rho_0 > 0$ is the *mass density in the reference initial configuration*, and r is the *external heat supply per unit of mass.*

Neglecting any hereditary contribution to mechanical dissipation, h is described by the following linearized constitutive equation (see [22]):

$$h(\boldsymbol{x},t) = \frac{\Theta_0}{\rho_0}\left[\mathbf{B}\cdot\mathbf{E}_t(\boldsymbol{x},t) + \frac{\rho_0 c}{\Theta_0}\,\vartheta_t(\boldsymbol{x},t) + \int_0^\infty a(\tau)\,\vartheta_t(\boldsymbol{x},t-\tau)\,d\tau\right]\,, \tag{2.6}$$

where $\mathbf{B}$ is a symmetric second order tensor, $a\colon \mathbb{R}^+ \to \mathbb{R}$ is the *energy memory kernel*, $c > 0$ is the *specific heat* of the body and $\cdot$ represents the tensorial scalar product.

According to previous constitutive equations, in [12] a mathematical model for a Mindlin-Timoshenko thermo-viscoelastic plate is derived, and a system of the type (1.1) is obtained.

Questions related to longtime behavior and stabilization of Timoshenko systems have attracted considerable attention in recent years. We start to recall some results about the Timoshenko beam without internal damping; subsequently, we consider some problems concerning Timoshenko beam with internal damping of memory type, and Timoshenko plate or with internal viscoelastic dissipation or with viscoelastic dissipation boundary condition.

By the energy method combined with the theory of semigroups, Kim and Renardy [20] investigate uniform stabilization of the Timoshenko beam with boundary control. Subsequently, Feng, Shi and Zhang [8] study the stabilization of vibrations in a Timoshenko beam by nonlinear boundary feedback. Analogously to the previous work, well-posedness is proved when the feedback operators are maximal monotone. Sufficient additional conditions are then given for energy to decay asymptotically or at a uniform algebraic or exponential rate.

Yan, Hou and Feng [26] obtain necessary and sufficient conditions for asymptotic stability and exponential energy decay for the Timoshenko beam equations

$$\rho\,\partial_{tt}w + \partial_x[K(\varphi - \partial_x w)] = 0, \quad I_\rho\,\partial_{tt}\varphi - \partial_x(EI\,\partial_x\varphi) + K(\varphi - \partial_x w) = 0,$$

with clamped left end.

Liu and Peng [23] study the exponential stability of a Timoshenko beam with viscoelastic damping of memory type. The corresponding model is given by

$$\rho\,\partial_{tt}u(t) - \partial_x\Big\{g_1(0)\,[\partial_x u(t) - \phi(t)] + \int_0^\infty g_1'(s)\,[\partial_x u(t-s) - \phi(t-s)]\;ds\Big\} = 0,$$

$$I_\rho\,\partial_{tt}\phi(t) - \partial_{xx}\left[g_2(0)\phi(t) + \int_0^\infty g_2'(s)\phi(t-s)\,ds\right] - \Big\{g_1(0)\,[\partial_x u(t) - \phi(t)] + \int_0^\infty g_1'(s)\,[\partial_x u(t-s) - \phi(t-s)]\,ds\Big\} = 0,$$

where g_i $(i = 1, 2)$ are relaxation functions which decay exponentially and have appropriate sign properties, producing the dissipation of the system. By contradiction arguments, involving the necessary and sufficient condition for a strongly continuous semigroup to be exponentially stable, the exponential stability is found.

Ammar-Khodja et al [1] study the rates of energy decay for a Timoshenko beam with internal damping, given by a functional in the bending history only, described by the following system

$$\begin{aligned} &\rho_1\,\partial_{tt}\varphi(t) - k\,\partial_x[\partial_x\varphi(t) + \psi(t)] = 0, \\ &\rho_2\,\partial_{tt}\psi(t) - b\,\partial_{xx}\psi(t) + \int_0^t g(t-s)\partial_{xx}\psi(s)\,ds \\ &\qquad\qquad\qquad\qquad\qquad + k[\partial_x\varphi(t) + \psi(t)] = 0, \end{aligned}$$

with homogeneous boundary conditions. The positive, decreasing memory kernel g belongs to $C^2[0,\infty)$, and $b - \int_0^\infty g(s)\,ds > 0$. First, the case $k/\rho_1 = b/\rho_2$ is considered. If g decays exponentially, then energy decays at a uniform exponential rate. Via energy integrals, if g decays like t^{-p} $(p > 2)$ (and is convex), then E decays uniformly and at least this fast. Subsequently, by approximating the memory kernel with exponential polynomials and analyzing the resulting problem by semigroup methods, if $k/\rho_1 \neq b/\rho_2$ and g decays exponentially, then energy does not decay at a uniform rate.

Giorgi and Vegni [14] investigate a mathematical model for viscoelastic beams, based on the Mindlin-Timoshenko assumptions, and derived in the framework of the well-established theory of linear viscoelasticity, according to the approximation procedure due to Lagnese [21] for the Kirchhoff viscoelastic beams and plates. Assuming a nonlinear body force acting on the beam, they show that this model generates a strongly continuous semigroup which acts on the appropriate phase space. The existence of an absorbing set for the solution of the problem is also studied. Furthermore, in [13] they investigate the longtime behavior of the mathematical model of a homogeneous viscoelastic plate based on the Mindlin-Timoshenko assumptions. Supposing that memory kernels decay exponentially, the exponentially decay of the energy is found. In this case, no thermal effects, acting on the plate, are considered.

In [15], Grasselli and Pata deal with a class of infinite-dimensional dissipative dynamical systems generated by evolution equations with linear memory terms and subject to time-dependent external forces. The longtime behavior of these systems is studied, and viscoelasticity and heat conduction with memory are considered as examples.

Moreover, Grasselli, Pata and Prouse [16] study the Timoshenko model of a viscoelastic beam consisting of two coupled second order linear integrodifferential hyperbolic equations with semilinear external forces acting on the beam.

The existence of a uniform attractor is showed, provided the semilinear external forces satisfy appropriate conditions.

De Lima Santos [4] analyses the solutions to a Timoshenko system of the form

$$\partial_{tt}u(t) - \Delta u(t) - \alpha \sum_{i=1}^{n} \partial_{x_i} v(t) + \beta u(t) = 0 \qquad \text{in } \Omega \times (0, \infty),$$

$$\partial_{tt}v(t) - \Delta v(t) + \alpha \sum_{i=1}^{n} \partial_{x_i} u(t) + f\left(v(t)\right) = 0 \quad \text{in } \Omega \times (0, \infty),$$

along with Dirichlet type conditions on Γ_0, one part of the boundary, and memory type boundary conditions on Γ_1, another part of the boundary. Under suitable assumptions, it is shown that exponential decay leads to exponential decay of the energy while polynomial decay of the kernels leads to polynomial decay. The technique of proof is based on the construction of Lyapunov functionals.

Muñoz Rivera and Oquendo [24] consider the Mindlin-Timoshenko plate with viscoelastic dissipation boundary conditions on part of the boundary. The energy of the system decays exponentially (respectively polynomially) to zero as time goes to infinity provided the relaxation functions appearing in the boundary conditions decay exponentially (respectively polynomially) to zero.

3. Hypotheses, Notations and Mathematic Tools

Given a Hilbert space $\mathcal{H}$, we denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$ the inner product and the norm on $\mathcal{H}$, respectively. We shall often be concerned with 2-dimensional vector functions with both components in $L^2(\Omega)$ or in $H^1(\Omega)$ or in another Hilbert space $\mathcal{H}$. We shall use the notation

$$\boldsymbol{L}^2(\Omega) = [L^2(\Omega)]^2, \quad \boldsymbol{H}^1(\Omega) = [H^1(\Omega)]^2, \quad \boldsymbol{\mathcal{H}} = \mathcal{H}^2.$$

The inner product and the norm on $\boldsymbol{L}^2(\Omega)$ or $L^2(\Omega)$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, without subscript.

Furthermore, we introduce the weighted L^2-spaces with respect to the measure $\alpha(s)ds$, endowed with the inner product on $\mathcal{H}$, as

$$L^2_\alpha(\mathbb{R}^+, \mathcal{H}) = \left\{ \phi : \mathbb{R}^+ \to \mathcal{H} : \|\phi\|^2_{L^2_\alpha(\mathbb{R}^+, \mathcal{H})} = \int_0^\infty \alpha(s) \|\phi\|^2_{\mathcal{H}} ds < \infty \right\}, \quad (3.1)$$

where $\alpha : \mathbb{R}^+ \to [0, \infty)$ is a given measurable function.

In order to give a precise formulation of our problem, on account of (1.2), for any $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$, we can set (see, e.g., [13])

$$\langle -\mathcal{A}\boldsymbol{v}, \boldsymbol{w}\rangle = \int_\Omega \left[\left(\frac{\partial v_1}{\partial x_1} + \nu_0 \frac{\partial v_2}{\partial x_2} \right) \frac{\partial w_1}{\partial x_1} + \frac{1-\nu_0}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \frac{\partial w_1}{\partial x_2} \right.$$
$$\left. + \frac{1-\nu_0}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \frac{\partial w_2}{\partial x_1} + \left(\nu_0 \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) \frac{\partial w_2}{\partial x_2} \right] d\Omega,$$

and in particular we have

$$\langle -\mathcal{A}\boldsymbol{v}, \boldsymbol{v}\rangle = \int_\Omega \left[\left| \frac{\partial v_1}{\partial x_1} \right|^2 + \left| \frac{\partial v_2}{\partial x_2} \right|^2 + 2\nu_0 \frac{\partial v_1}{\partial x_1} \frac{\partial v_2}{\partial x_2} \right.$$
$$\left. + \frac{1-\nu_0}{2} \left| \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right|^2 \right] d\Omega.$$

Then, we introduce the Hilbert space

$$\mathcal{V} = \left\{ \boldsymbol{v} = (v_1, v_2) \in \boldsymbol{L}^2(\Omega) : \ \frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \in L^2(\Omega) \right\},$$

with the norm

$$\|\boldsymbol{v}\|^2_{\mathcal{V}} = \int_\Omega \left[v_1 + v_2 + \left| \frac{\partial v_1}{\partial x_1} \right|^2 + \left| \frac{\partial v_2}{\partial x_2} \right|^2 + \left| \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right|^2 \right] d\Omega.$$

The kernel of $-\mathcal{A}$ is a subspace of $\mathcal{V}$ of the linear function of the type

$$\begin{bmatrix} c_0 x_2 + c_1 \\ -c_0 x_1 + c_2 \end{bmatrix},$$

where c_0, c_1, c_2 are constants. Afterwards, we denote by $V_{\mathcal{A}}$ the linear subspace of $\mathcal{V}$ which is orthogonal in $\boldsymbol{L}^2(\Omega)$ to the kernel of $-\mathcal{A}$, and is characterized by the conditions

$$\int_\Omega (x_2 v_1 - x_1 v_2) d\Omega = 0, \quad \int_\Omega v_1 \, d\Omega = 0, \quad \int_\Omega v_2 \, d\Omega = 0.$$

Conditions on Memory Kernels. We assume

$$\nu,\ \mu,\ \kappa,\ \gamma \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{H.0}$$

such that for any $s \in \mathbb{R}^+$

$$\nu(s) \geq 0, \qquad \mu(s) \geq 0, \qquad \kappa(s) \geq 0, \qquad \gamma(s) \geq 0, \tag{H.1}$$

$$\nu'(s) \leq 0, \qquad \mu'(s) \leq 0, \qquad \kappa'(s) \leq 0, \qquad \gamma'(s) \leq 0, \tag{H.2}$$

hold. In view of (H.0)-(H.1), we define

$$\mu_0 = \int_0^\infty \mu(s)ds,\ \nu_0 = \int_0^\infty \nu(s)ds,\ \kappa_0 = \int_0^\infty \kappa(s)ds,\ \gamma_0 = \int_0^\infty \gamma(s)ds.$$

Besides, for every $s \in \mathbb{R}^+$ and some $\delta_\mu, \delta_\nu, \delta_\kappa, \delta_\gamma > 0$

$$\begin{aligned} &\mu'(s) + \delta_\mu \mu(s) \leq 0, \quad \nu'(s) + \delta_\nu \nu(s) \leq 0, \\ &\kappa'(s) + \delta_\kappa \kappa(s) \leq 0, \quad \gamma'(s) + \delta_\gamma \gamma(s) \leq 0, \end{aligned} \tag{H.3}$$

are assumed. Let $\delta = \min\{\delta_\mu, \delta_\nu, \delta_\kappa, \delta_\gamma\}$. There exists also $s_\mu, s_\nu > 0$ such that $\mu' \in L^2(0, s_\mu)$ and $\nu' \in L^2(0, s_\nu)$

$$\mu'(s) + M_\mu \mu(s) \geq 0, \qquad \nu'(s) + M_\nu \nu(s) \geq 0, \tag{H.4}$$

for every $s > s_\mu$ and $s > s_\nu$, respectively, and some $M_\mu, M_\nu > 0$. Finally, let μ be suitably dominated by ν, namely, for any $s \in \mathbb{R}^+$

$$\mu(s) \leq \nu(s). \tag{H.5}$$

Remark 3.1. Assumption (H.3) implies the exponential decay of the kernels. This hypothesis seems unavoidable in order to have exponential decay of the associated linear problem and it is standard assumed (cf., e.g., [6, 23]). On the other hand, it seems quite obvious that to have exponential decay of the energy, the kernels must show the same rate of decay (see [11]).

Conditions on External Sources. For $i = 1, 2, 3, 4$

$$f_i \in L^1(\mathbb{R}^+, L^2(\Omega)). \tag{F.0}$$

Hilbert Spaces. Finally, we consider the Hilbert spaces

$$\begin{aligned} &H = L^2(\Omega), && V = H_0^1(\Omega), \\ &H_\nu = L^2_\nu(\mathbb{R}^+, H), && V_\nu = L^2_\nu(\mathbb{R}^+, V), \quad V_{\nu\mathcal{A}} = L^2_\nu(\mathbb{R}^+, V_\mathcal{A}), \\ &H_\mu = L^2_\nu(\mathbb{R}^+, H), && V_\mu = L^2_\mu(\mathbb{R}^+, V), \\ &H_\gamma = L^2_\gamma(\mathbb{R}^+, H), && V_\gamma = L^2_\gamma(\mathbb{R}^+, V), \\ &H_\kappa = L^2_\kappa(\mathbb{R}^+, H), && V_\kappa = L^2_\nu(\mathbb{R}^+, V). \end{aligned}$$

Introduce also

$$Z = V_\mathcal{A} \times \boldsymbol{H} \times V \times H \times H \times (V_{\nu\mathcal{A}} \cap \boldsymbol{H}_\mu) \times V_\mu \times (V_\kappa \cap H_\gamma).$$

Remark 3.2. In account of Korn inequality (see, e.g., [5, 9]), it is possible to prove [13] that the operator $-\mathcal{A}$ in V is coercive and consequently that it defines a norm which is equivalent to the usual norm in $\boldsymbol{H}^1(\Omega)$. We shall often exploit this fact during calculations.

Remark 3.3. Since the model we study is quite heavy, to lighten the notation and help the reader to follow calculations, we will not stress the explicit dependance on the constants arising in inequalities due to equivalence between norms. Indeed, we shall use different norms in Euclidean spaces, and, on account of Remark 3.2, different norms in V.

4. Well-Posedness

Theorem 4.1. *Assume conditions* (H.0)-(H.2) *and* (F.0) *hold true. Suppose also*

$$(\boldsymbol{v}_0, \boldsymbol{v}_1, u_0, u_1, \theta_0, \boldsymbol{\zeta}_0, \omega_0, \eta_0) \in Z.$$

Then, for any $T > 0$, system (1.10)-(1.17) *has a unique solution on* $[0, T]$

$$(\boldsymbol{v}, \boldsymbol{v}_t, u, u_t, \theta, \boldsymbol{\zeta}, \omega, \eta) \in C^0([0, T], Z).$$

Furthermore, suppose that $(\boldsymbol{v}_i, \boldsymbol{v}_{ti}, u_i, u_{ti}, \theta_i, \boldsymbol{\zeta}_i^t, \omega_i^t, \eta_i^t)$, are solutions corresponding to initial data $(\boldsymbol{v}_{0i}, \boldsymbol{v}_{1i}, u_{0i}, u_{1i}, \theta_{0i}, \boldsymbol{\zeta}_{0i}, \omega_{0i}, \eta_{0i})$, and external sources f_{j_i}, $j = 1, 2, 3,\ 4$, $i = 1, 2$. Then, there exists an increasing function $\Phi : [0, \infty) \to [0, \infty)$ such that whenever

$$\|(\boldsymbol{v}_{0i}, \boldsymbol{v}_{1i}, u_{0i}, u_{1i}, \theta_{0i}, \boldsymbol{\zeta}_{0i}, \omega_{0i}, \eta_{0i})\|_Z + \sum_{j=1}^{4} \|f_{j_i}\|_{L^1([0,T],H)} \leq M,$$

for some $M \geq 0$, it is verified that

$$\begin{aligned}
&\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_{V_{\nu\mathcal{A}}} + \|\boldsymbol{v}_{t1}(t) - \boldsymbol{v}_{t2}(t)\| + \|u_1(t) - u_2(t)\|_V \\
&\quad + \|u_{t1}(t) - u_{t2}(t)\| + \|\theta_1(t) - \theta_2(t)\| + \|\boldsymbol{\zeta}_1^t - \boldsymbol{\zeta}_2^t\|_{V_{\nu\mathcal{A}}} \\
&\quad + \|\boldsymbol{\zeta}_1^t - \boldsymbol{\zeta}_2^t\|_{\boldsymbol{H}_\mu} + \|\omega_1^t - \omega_2^t\|_{V_\mu} + \|\eta_1^t - \eta_2^t\|_{V_\kappa} + \|\eta_1^t - \eta_2^t\|_{H_\gamma} \\
&\leq \Phi(M)\, (\|\boldsymbol{v}_{01} - \boldsymbol{v}_{02}\|_{V_{\nu\mathcal{A}}} + \|\boldsymbol{v}_{11} - \boldsymbol{v}_{12}\| + \|u_{01} - u_{02}\|_V \\
&\quad + \|u_{11} - u_{12}\| + \|\theta_{01} - \theta_{02}\| + \|\boldsymbol{\zeta}_{01} - \boldsymbol{\zeta}_{02}\|_{V_{\nu\mathcal{A}}} \\
&\quad + \|\boldsymbol{\zeta}_{01} - \boldsymbol{\zeta}_{02}\|_{\boldsymbol{H}_\mu} \|\omega_{01} - \omega_{02}\|_{V_\mu} \\
&\quad + \|\eta_{01} - \eta_{02}\|_{V_\kappa} + \|\eta_{01} - \eta_{02}\|_{H_\gamma} \\
&\quad + \|f_{11} - f_{12}\|_{L^1([0,T],H)} + \|f_{21} - f_{22}\|_{L^1([0,T],H)}
\end{aligned}$$

$$+\|f_{31}-f_{32}\|_{L^1([0,T],H)}+\|f_{41}-f_{42}\|_{L^1([0,T],H)}\Big). \tag{4.1}$$

The proof of the above theorem is omitted. It can be carried out via Faedo-Galerkin method (with due technical modifications, see for instance [14]).

Let us denote the solution

$$z(t)=\left[\begin{array}{cccccccc} \boldsymbol{v}(t) & \boldsymbol{v}_t(t) & u(t) & u_t(t) & \theta(t) & \boldsymbol{\zeta}^t & \omega^t & \eta^t \end{array}\right],$$

with initial data $z(0)=z_0\in Z$ by $S(t)z_0$. Hence, on account of Theorem 4.1, the solution is described by the continuous semigroup $S(t)$ acting on the space Z, i.e. $S(t)$ enjoys the following properties:

(i) $S(t):Z\to Z$, continuous for every $t\geq 0$,
(ii) $S(0)=\mathbb{I}$ (identity on Z),
(iii) $\lim\limits_{t\to 0^+} S(t)z=z$ for every $z\in Z$,
(iv) $S(t)S(\tau)=S(t+\tau)$, for every $t,\tau\geq 0$.

5. The Main Result

Let us introduce the energy of system (1.10)-(1.17)

$$\begin{aligned}\mathcal{E}(t)=&\|\boldsymbol{v}(t)\|^2_{V_{\mathcal{A}}}+\|\partial_t\boldsymbol{v}(t)\|^2+\|\partial_t u(t)\|^2+\|\theta(t)\|^2\\ &+\|\nabla u(t)+\boldsymbol{v}(t)\|^2+\|\boldsymbol{\zeta}^t\|^2_{V_{\nu\mathcal{A}}}+\|\nabla\omega^t+\boldsymbol{\zeta}^t\|^2_{\boldsymbol{H}_\mu}+\|\eta^t\|^2_{V_\kappa\cap H_\gamma}.\end{aligned} \tag{5.1}$$

Theorem 5.1. *Assume conditions* (H.0)-(H.4) *and* (F.0) *hold true. Then there exist* $\epsilon>0$ *and two positive constants* C_1, C_2 *such that, for every* $t\geq 0$, *the following estimate*

$$\mathcal{E}(t)\leq C_1\mathcal{E}(0)e^{-\epsilon t}+\bigg\{C_2\int_0^t e^{-\epsilon/2(t-\tau)}\left[\|f_1(\tau)\|+\|f_2(\tau)\|\right.$$
$$\left.+\|f_3(\tau)\|+\|f_4(\tau)\|\right]d\tau\bigg\}^2$$

holds.

We can state also the following corollary.

Corollary 5.2. *If, for a positive real constant* K

$$\sup_{\sigma\geq 0}\int_\sigma^{\sigma+1}\left[\|f_1(\tau)\|+\|f_2(\tau)\|+\|f_3(\tau)\|+\|f_4(\tau)\|\right]d\tau\leq K$$

holds, then every ball $\mathcal{B}_0$ of Z centered at zero and of radius greater than $C_2 = C_2(K)$ is an absorbing set[1] *for the semigroup $S(t)$ generated by the system* (1.10)-(1.17).

6. Proof of Theorem 5.1

In the sequel, we will denote by $C > 0$ a generic constant, which may vary even within the same formula.

We start from equation (1.10) and multiply it in $\boldsymbol{H}$ by $\partial_t \boldsymbol{v}$, to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_t \boldsymbol{v}\|^2 + \|\boldsymbol{v}\|^2_{V_{\mathcal{A}}}\right) + \langle \nabla u + \boldsymbol{v}, \partial_t \boldsymbol{v}\rangle + \langle \nabla\theta, \partial_t \boldsymbol{v}\rangle - \int_0^\infty \nu(s)\langle \mathcal{A}\boldsymbol{\zeta}(s), \partial_t \boldsymbol{v}\rangle ds + \int_0^\infty \mu(s)\langle \nabla\omega(s) + \boldsymbol{\zeta}(s), \partial_t \boldsymbol{v}\rangle ds = \langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \partial_t \boldsymbol{v}\rangle.$$

We consider equation (1.11) multiplied in H by $\partial_t u$. Performing some integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|^2 + \langle \nabla u + \boldsymbol{v}, \partial_t \nabla u\rangle + \int_0^\infty \mu(s)\langle \nabla\omega(s) + \boldsymbol{\zeta}(s), \partial_t \nabla u\rangle = \langle f_3, \partial_t u\rangle.$$

We consider equation (1.12), and multiply it in H by θ. After integration by parts, we end

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \|\theta\|^2 - \langle \partial_t \boldsymbol{v}, \nabla\theta\rangle + \int_0^\infty \kappa(s)\langle \nabla\eta(s), \nabla\theta\rangle ds + \iota_0 \int_0^\infty \gamma(s)\langle \eta(s), \theta\rangle ds = \langle f_4, \theta\rangle.$$

By adding the three previous equations we have got, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_t \boldsymbol{v}\|^2 + \|\boldsymbol{v}\|^2_{V_{\mathcal{A}}} + \|\partial_t u\|^2 + \|\theta\|^2 + \|\nabla u + \boldsymbol{v}\|^2\right) + \|\theta\|^2$$

[1] A set $\mathcal{B}_0 \subset Z$ is said to be *absorbing* for the semigroup $\{S(t)\}$ if for any bounded set $\mathcal{B} \subset Z$ there exists $t_{\mathcal{B}} \geq 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}_0$ for every $t \geq t_{\mathcal{B}}$.

$$
\begin{aligned}
&- \int_0^\infty \nu(s) \langle \mathcal{A}\boldsymbol{\zeta}(s), \partial_t \boldsymbol{v} \rangle ds + \int_0^\infty \mu(s) \langle \nabla\omega(s) + \boldsymbol{\zeta}(s), \partial_t \boldsymbol{v} + \partial_t \nabla u \rangle ds \\
&+ \int_0^\infty \kappa(s) \langle \nabla\eta(s), \nabla\theta \rangle ds + \iota_0 \int_0^\infty \gamma(s) \langle \eta(s), \theta \rangle ds = \langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \partial_t \boldsymbol{v} \rangle \\
&+ \langle f_3, \partial_t u \rangle + \langle f_4, \theta \rangle. \qquad (6.1)
\end{aligned}
$$

We proceed by performing some estimates on the equations governing the memory terms. We consider (1.13) and multiply it in $V_{\nu\mathcal{A}}$ by $\boldsymbol{\zeta}$, obtaining

$$
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}} + \langle \partial_s \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle_{V_{\nu\mathcal{A}}} = \langle \boldsymbol{\zeta}, \partial_t \boldsymbol{v} \rangle_{V_{\nu\mathcal{A}}}. \qquad (6.2)
$$

We add equation (1.13) to the gradient of (1.14), to have

$$
\partial_t(\boldsymbol{\zeta} + \nabla\omega) + \partial_s(\boldsymbol{\zeta} + \nabla\omega) = \partial_t(\boldsymbol{v} + \nabla u).
$$

We multiply this equation by $(\boldsymbol{\zeta} + \nabla\omega)$ in H_μ, and get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta} + \nabla\omega\|^2_{\boldsymbol{H}_\mu} &+ \langle \partial_s(\boldsymbol{\zeta} + \nabla\omega), \boldsymbol{\zeta} + \nabla\omega \rangle_{\boldsymbol{H}_\mu} \\
&= \langle \partial_t(\boldsymbol{v} + \nabla u), \boldsymbol{\zeta} + \nabla\omega \rangle_{\boldsymbol{H}_\mu}. \qquad (6.3)
\end{aligned}
$$

We consider equation (1.15), and multiply it by η in $V_\kappa \cap H_\gamma$; this yields to

$$
\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{V_\kappa \cap H_\gamma} + \langle \partial_s \eta, \eta \rangle_{V_\kappa \cap H_\gamma} = \langle \eta, \theta \rangle_{V_\kappa \cap H_\gamma}. \qquad (6.4)
$$

Observe that (H.3) gives us an estimate on the term $\langle \partial_s \boldsymbol{\zeta}(s), \boldsymbol{\zeta}(s) \rangle_{V_\nu}$ in equation (6.2). Indeed by integration by parts we have

$$
\langle \partial_s \boldsymbol{\zeta}(s), \boldsymbol{\zeta}(s) \rangle_{V_{\nu\mathcal{A}}} \geq \frac{\delta}{2}\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}}.
$$

In account of (H.3), this estimate may be repeated on the similar terms appearing in equations (6.3) and (6.4). Hence, adding (6.2)-(6.4), we obtain:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}} + \|\boldsymbol{\zeta} + \nabla\omega\|^2_{\boldsymbol{H}_\mu} + \|\eta\|^2_{V_\kappa \cap H_\gamma}\right) \\
&+ \frac{\delta}{2}\left(\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}} + \|\boldsymbol{\zeta} + \nabla\omega\|^2_{\boldsymbol{H}_\mu} + \|\eta\|^2_{V_\kappa \cap H_\gamma}\right) \\
&\leq \langle \boldsymbol{\zeta}, \partial_t \boldsymbol{v} \rangle_{V_{\nu\mathcal{A}}} + \langle \partial_t(\boldsymbol{v} + \nabla u), \boldsymbol{\zeta} + \nabla\omega \rangle_{\boldsymbol{H}_\mu} + \langle \eta, \theta \rangle_{V_\kappa \cap H_\gamma}. \qquad (6.5)
\end{aligned}
$$

We add equation (6.1) to (6.5), and have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(&\|\partial_t\boldsymbol{v}\|^2+\|\boldsymbol{v}\|^2_{V_{\mathcal{A}}}+\|\partial_t u\|^2+\|\theta\|^2+\|\nabla u+\boldsymbol{v}\|^2_H\\&+\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}}+\|\boldsymbol{\zeta}+\nabla\omega\|^2_{\boldsymbol{H}_\mu}+\|\eta\|^2_{V_\kappa\cap H_\gamma}\Big)\\&+\|\theta\|^2_H+\frac{\delta}{2}\left(\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}}+\|\boldsymbol{\zeta}+\nabla\omega\|^2_{\boldsymbol{H}_\mu}+\|\eta\|^2_{V_\kappa\cap H_\gamma}\right)\\&\le\langle\begin{bmatrix}f_1\\f_2\end{bmatrix},\partial_t\boldsymbol{v}\rangle+\langle f_3,\partial_t u\rangle+\langle f_4,\theta\rangle.\end{aligned}\tag{6.6}$$

Remark 6.1. We consider here the case $\iota_0=1$. The case $\iota_0=-1$ can also be dealt with minor modifications. We just point out that to treat the term $2\int_0^\infty\gamma(s)(\eta(s),\theta\rangle ds$ it is necessary to have the memory kernel γ to be suitably dominated by κ: $\gamma(\sigma)\le\delta_0\kappa(\sigma)$ and $\kappa'(\sigma)+\delta\kappa(\sigma)\le 0$ for all $\sigma\in\mathbb{R}^+$ for some $0<\delta_0<\delta$ (see [17]).

Equation (6.6) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}\le\left(\|f_1\|+\|f_2\|+\|f_3\|+\|f_4\|\right)\mathcal{E}^{1/2}\tag{6.7}$$

and, using (F.0) and a Gronwall type Lemma (see, e.g. Lemma 2.4 in [17]), we deduce that $\mathcal{E}$ is bounded over $\mathbb{R}^+$.

We consider the product in H of equation (1.10) by $\boldsymbol{v}$; after performing some integration by parts, we have

$$\begin{aligned}\langle\partial_{tt}\boldsymbol{v},\boldsymbol{v}\rangle+\|\boldsymbol{v}\|^2_{V_{\mathcal{A}}}+\langle\nabla\theta,\boldsymbol{v}\rangle+\langle\nabla u+\boldsymbol{v},\boldsymbol{v}\rangle+\langle\boldsymbol{\zeta},\boldsymbol{v}\rangle_{V_{\nu\mathcal{A}}}&\\+\langle\nabla\omega+\boldsymbol{\zeta},\boldsymbol{v}\rangle_{\boldsymbol{H}_\mu}=\langle\begin{bmatrix}f_1\\f_2\end{bmatrix},\boldsymbol{v}\rangle.&\end{aligned}\tag{6.8}$$

Multiplication of equation (1.11) by u in H, and some integration by parts lead to

$$(\partial_{tt}u,u\rangle+(\nabla u+\boldsymbol{v},\nabla u\rangle+(\nabla\omega+\boldsymbol{\zeta},\nabla u)_{\boldsymbol{H}_\mu}=(f_3,u\rangle.$$

Adding last two equations, we have

$$\begin{aligned}\langle\partial_{tt}\boldsymbol{v},\boldsymbol{v}\rangle+\langle\partial_{tt}u,u\rangle+\|\boldsymbol{v}\|^2_{V_{\mathcal{A}}}+\langle\nabla\theta,\boldsymbol{v}\rangle+\|\nabla u+\boldsymbol{v}\|^2&\\+\langle\boldsymbol{\zeta},\boldsymbol{v}\rangle_{V_{\nu\mathcal{A}}}+\langle\nabla\omega+\boldsymbol{\zeta},\nabla u+\boldsymbol{v}\rangle_{\boldsymbol{H}_\mu}=\langle\begin{bmatrix}f_1\\f_2\end{bmatrix},\boldsymbol{v}\rangle+\langle f_3,u\rangle.&\end{aligned}$$

Integrating by parts and using Young inequality, we have

$$\begin{aligned}\langle \partial_{tt}\boldsymbol{v},\boldsymbol{v}\rangle + \langle \partial_{tt}u,u\rangle + \frac{1}{6}\|\boldsymbol{v}\|^2_{V_{\mathcal{A}}} + \frac{1}{2}\|\nabla u+\boldsymbol{v}\|^2_H \\ \le C\left(\|\nabla\omega+\boldsymbol{\zeta}\|^2_{\boldsymbol{H}_\mu} + \|\boldsymbol{\zeta}\|^2_{V_\nu\mathcal{A}} + \|\theta\|^2_H\right) \\ + \langle \begin{bmatrix} f_1 \\ f_2\end{bmatrix}, \boldsymbol{v}\rangle + \langle f_3,u\rangle. \quad (6.9)\end{aligned}$$

We introduce the functionals

$$\mathcal{L}_1 = -\langle \partial_t\boldsymbol{v}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle, \quad (6.10)$$

$$\mathcal{L}_2 = -\langle \partial_t u, \int_0^\infty \mu(s)\omega(s)ds\rangle, \quad (6.11)$$

and study the behavior of their derivatives with respect to the time. In account of equations (1.10) and (1.13), we have

$$\begin{aligned}\partial_t\mathcal{L}_1 =& -\langle \partial_{tt}\boldsymbol{v}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle - \langle \partial_t\boldsymbol{v}, \int_0^\infty \nu(s)\partial_t\boldsymbol{\zeta}(s)ds\rangle \\ =& -\langle \mathcal{A}\boldsymbol{v}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle + \langle \nabla u+\boldsymbol{v}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle \\ &+ \langle \nabla\theta, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle - \langle \int_0^\infty \nu(s)\mathcal{A}\boldsymbol{\zeta}(s)ds, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle \\ &+ \langle \int_0^\infty \mu(s)[\nabla\omega(s)+\boldsymbol{\zeta}(s)]ds, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle - \nu_0\|\partial_t\boldsymbol{v}\|^2_H \\ &- \langle \begin{bmatrix} f_1 \\ f_2\end{bmatrix}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle + \langle \partial_t\boldsymbol{v}, \int_0^\infty \nu(s)\partial_s\boldsymbol{\zeta}(s)ds\rangle. \quad (6.12)\end{aligned}$$

We notice that for every function $u \in H_\nu$ and $v \in H_\mu$, using Schwartz inequality in $L^2(\mathbb{R}^+)$ and then in H, we have

$$\begin{aligned}&\langle \int_0^\infty \nu(s)u(s)ds, \int_0^\infty \mu(s)v(s)ds\rangle \\ &\le \sqrt{\nu_0\mu_0}\langle \left[\int_0^\infty \nu(s)u^2(s)ds\right]^{1/2}, \left[\int_0^\infty \mu(s)v^2(s)ds\right]^{1/2}\rangle \\ &\le \sqrt{\nu_0\mu_0}\,\|u\|_{H_\nu}\|v\|_{H\mu}. \quad (6.13)\end{aligned}$$

Now, we devote some effort to estimate the term appearing in the right hand side of equation (6.12). After integration by parts, exploiting (6.13) and using

Young inequality, we obtain

$$-\langle \mathcal{A}\boldsymbol{v}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle \le \epsilon\|\boldsymbol{v}\|^2_{V_{\mathcal{A}}} + C(\epsilon)\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}}, \tag{6.14}$$

$$\langle \nabla u + \boldsymbol{v}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle \le \epsilon\|\nabla u + \boldsymbol{v}\|^2_H + C(\epsilon)\|\boldsymbol{\zeta}\|^2_{H_\nu}, \tag{6.15}$$

$$-\langle \int_0^\infty \nu(s)\mathcal{A}\boldsymbol{\zeta}(s)ds, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle \le \nu_0\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}}, \tag{6.16}$$

$$\begin{aligned}\langle \int_0^\infty \mu(s)[\nabla\omega(s) + \boldsymbol{\zeta}(s)]ds, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle \\ \le \frac{\mu_0}{2}\|\nabla\omega + \boldsymbol{\zeta}\|^2_{\boldsymbol{H}_\mu} + \frac{\nu_0}{2}\|\boldsymbol{\zeta}\|^2_{H_\nu},\end{aligned} \tag{6.17}$$

$$\begin{aligned}\langle \nabla\theta, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle &= -\langle \theta, \int_0^\infty \nu(s)\nabla\cdot\boldsymbol{\zeta}(s)ds\rangle \\ &\le \sqrt{\nu_0}\,\|\theta\|_H\|\nabla\cdot\boldsymbol{\zeta}\|_{H_\nu} \le \epsilon\|\theta\|^2_H + C(\epsilon)\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}},\end{aligned} \tag{6.18}$$

where $\epsilon > 0$ and $C(\epsilon) > 0$ appear in Young inequality, and ϵ will be fixed small later.

Recalling [11], in view of (H.4), we get

$$\begin{aligned}-\nu_0\|\partial_t\boldsymbol{v}\|^2_H + \langle \partial_t\boldsymbol{v}, \int_0^\infty \nu(s)\partial_s\boldsymbol{\zeta}(s)ds\rangle \\ \le -\frac{\nu_0}{2}\|\partial_t\boldsymbol{v}\|^2_H + C\|\boldsymbol{\zeta}\|^2_{H_\nu}.\end{aligned} \tag{6.19}$$

Collecting inequalities (6.14)-(6.19) in equation (6.12), we have

$$\begin{aligned}\partial_t\mathcal{L}_1 + \frac{\nu_0}{2}\|\partial_t\boldsymbol{v}\|^2 \le\ & \epsilon(\|\boldsymbol{v}\|^2_{V_{\mathcal{A}}} + \|\nabla u + \boldsymbol{v}\|^2 + \|\theta\|^2) \\ & + C(\epsilon)(\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}} + \|\nabla\omega + \boldsymbol{\zeta}\|^2_{\boldsymbol{H}_\mu}) - \langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle.\end{aligned} \tag{6.20}$$

Following the same path, we estimate the time derivative of $\mathcal{L}_2$. Using equation (1.11) and (1.14) together with some integration by parts, we obtain

$$\begin{aligned}\partial_t\mathcal{L}_2 =& -\langle \partial_{tt}u, \int_0^\infty \mu(s)\omega(s)ds\rangle - \langle \partial_t u, \int_0^\infty \mu(s)\partial_t\omega(s)ds\rangle \\ =& \langle \boldsymbol{v} + \nabla u, \int_0^\infty \mu(s)\nabla\omega(s)ds\rangle - \mu_0\|\partial_t u\|^2\end{aligned}$$

$$+\langle \int_0^\infty \mu(s)(\boldsymbol{\zeta}(s)+\nabla\omega(s))ds, \int_0^\infty \mu(s)\nabla\omega(s)ds\rangle$$
$$+\langle \partial_t u, \int_0^\infty \mu(s)\partial_s\omega(s)ds\rangle - \langle f_3, \int_0^\infty \mu(s)\omega(s)ds\rangle. \tag{6.21}$$

We estimate the right hand side terms. Recall (6.13) and (6.19), we have

$$\langle \boldsymbol{v}+\nabla u, \int_0^\infty \mu(s)\nabla\omega(s)ds\rangle \le \epsilon\|\boldsymbol{v}+\nabla u\|^2 + C(\epsilon)\|\nabla\omega\|^2_{\boldsymbol{H}_\mu}, \tag{6.22}$$

$$-\mu_0\|\partial_t u\|^2 + \langle \partial_t u, \int_0^\infty \mu(s)\partial_s\omega(s)ds\rangle$$
$$\le -\frac{\mu_0}{2}\|\partial_t u\|^2 + C\|\nabla\omega\|^2_{\boldsymbol{H}_\mu}, \tag{6.23}$$

$$\langle \int_0^\infty \mu(s)[\boldsymbol{\zeta}(s)+\nabla\omega(s)]ds, \int_0^\infty \mu(s)\nabla\omega(s)ds\rangle$$
$$\le \frac{\mu_0}{2}\|\boldsymbol{\zeta}+\nabla\omega\|^2_{\boldsymbol{H}_\mu} + \frac{\mu_0}{2}\|\nabla\omega\|^2_{\boldsymbol{H}_\mu}. \tag{6.24}$$

Using the estimates (6.22)-(6.24) in equation (6.21), we have

$$\partial_t\mathcal{L}_2 + \frac{\mu_0}{2}\|\partial_t u\|^2 \le \epsilon\|\boldsymbol{v}+\nabla u\|^2$$
$$+ C(\epsilon)(\|\nabla\omega\|^2_{\boldsymbol{H}_\mu} + \|\boldsymbol{\zeta}+\nabla\omega\|^2_{\boldsymbol{H}_\mu}) - \langle f_3, \int_0^\infty \mu(s)\omega(s)ds\rangle, \tag{6.25}$$

and recalling (H.5), we easily obtain

$$\partial_t\mathcal{L}_2 + \frac{\mu_0}{2}\|\partial_t u\|^2 \le \epsilon\|\boldsymbol{v}+\nabla u\|^2$$
$$+ C(\epsilon)(\|\boldsymbol{\zeta}\|^2_{V_\nu\mathcal{A}} + \|\boldsymbol{\zeta}+\nabla\omega\|^2_{\boldsymbol{H}_\mu}) - \langle f_3, \int_0^\infty \mu(s)\omega(s)ds\rangle. \tag{6.26}$$

Now, we add (6.20) to (6.26), with a suitable choice of the parameter ϵ, we obtain

$$\partial_t(\mathcal{L}_1+\mathcal{L}_2) + \frac{\nu_0}{2}\|\partial_t\boldsymbol{v}\|^2 + \frac{\mu_0}{2}\|\partial_t u\|^2 \le \epsilon(\|\boldsymbol{v}\|^2_{V_\mathcal{A}} + \|\boldsymbol{v}+\nabla u\|^2 + \|\theta\|^2_H)$$
$$+ C(\epsilon)(\|\boldsymbol{\zeta}+\nabla\omega\|^2_{\boldsymbol{H}_\mu} + \|\boldsymbol{\zeta}\|^2_{V_\nu\mathcal{A}})$$
$$- \langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle - \langle f_3, \int_0^\infty \mu(s)\omega(s)ds\rangle. \tag{6.27}$$

Let $\overline{\alpha}$ and $\overline{\overline{\alpha}}$ be two positive constants, which will be fixed later; we add (6.9) multiplied by $\overline{\alpha}$ to (6.27) multiplied by $\overline{\overline{\alpha}}$ to (6.6). Using the trivial identity

$\langle \partial_{tt} u, u\rangle = \partial_t \langle \partial_t u, u\rangle - \|\partial_t u\|^2$, we obtain

$$\begin{aligned}
&\partial_t(\mathcal{E} + \overline{\overline{\alpha}}\mathcal{L}_1 + \overline{\overline{\alpha}}\mathcal{L}_2 + \overline{\alpha}\langle \partial_t \boldsymbol{v}, \boldsymbol{v}\rangle + \overline{\alpha}\langle \partial_t u, u\rangle) + (1 - \overline{\overline{\alpha}}\epsilon - \overline{\alpha}C)\|\theta\|_H^2 \\
&\quad + \left[\frac{\delta}{2} - \overline{\overline{\alpha}}C(\epsilon) - \overline{\alpha}C\right](\|\boldsymbol{\zeta}\|^2_{V_{\nu\mathcal{A}}} + \|\boldsymbol{\zeta} + \nabla\omega\|^2_{\boldsymbol{H}_\mu}) + \frac{\delta}{2}\|\eta\|^2_{V_\kappa \cap H_\gamma} \\
&\quad + \left(\frac{\overline{\overline{\alpha}}\nu}{2} - \overline{\alpha}\right)\|\partial_t \boldsymbol{v}\|^2 + \left(\frac{\overline{\overline{\alpha}}\mu}{2} - \overline{\alpha}\right)\|\partial_t u\|^2 \\
&\quad + \left(\frac{\overline{\alpha}}{2} - \overline{\overline{\alpha}}\epsilon\right)\|\boldsymbol{v} + \nabla u\|^2 + \left(\frac{\overline{\alpha}}{6} - \overline{\overline{\alpha}}\epsilon\right)\|\boldsymbol{v}\|^2_{V_\mathcal{A}} \\
&\leq \overline{\alpha}\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \boldsymbol{v}\rangle + \overline{\alpha}\langle f_3, u\rangle + \langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \partial_t \boldsymbol{v}\rangle + \langle f_3, \partial_t u\rangle + \langle f_4, \theta\rangle \\
&\quad - \overline{\overline{\alpha}}\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \int_0^\infty \nu(s)\boldsymbol{\zeta}(s)ds\rangle - \overline{\overline{\alpha}}\langle f_3, \int_0^\infty \mu(s)\omega(s)ds\rangle. \qquad (6.28)
\end{aligned}$$

We observe that the constants appearing in the left hand side of equation (6.28) are compatible; indeed, we want to guarantee that

$$1 - \overline{\overline{\alpha}}\epsilon - \overline{\alpha}C > 0, \qquad (6.29)$$

$$\frac{\delta}{2} - \overline{\overline{\alpha}}C(\epsilon) - \overline{\alpha}C > 0, \qquad (6.30)$$

$$\frac{\overline{\overline{\alpha}}\nu}{2} - \overline{\alpha} > 0, \qquad (6.31)$$

$$\frac{\overline{\overline{\alpha}}\mu}{2} - \overline{\alpha} > 0, \qquad (6.32)$$

$$\frac{\overline{\alpha}}{2} - \overline{\overline{\alpha}}\epsilon > 0, \qquad (6.33)$$

$$\frac{\overline{\alpha}}{6} - \overline{\overline{\alpha}}\epsilon > 0, \qquad (6.34)$$

and it is clear that (6.33) holds when (6.34) is satisfied. We verify that (6.34) is compatible with (6.31) and (6.32), if $\epsilon < \nu/12$ and $\epsilon < \mu/12$, respectively. Hence, for instance, we choose $\epsilon = \beta\mu < \min\{\nu/12, \mu/12\} = \mu/12$, with $0 < \beta < 1/12$, so to fix $C^* = C(\epsilon)$. Now, (6.29)-(6.30) are satisfied when

$$\overline{\overline{\alpha}} < \min\left\{\frac{1}{\beta\mu}(1 - \overline{\alpha}C), \frac{1}{C^*}\left(\frac{\delta}{2} - \overline{\alpha}C\right)\right\}, \qquad (6.35)$$

with

$$\overline{\alpha} < \frac{1}{C}\min\left\{1, \frac{\delta}{2}\right\}.$$

From (6.31)-(6.32) and (H.5), we find $\overline{\overline{\alpha}} > (2/\mu)\overline{\alpha}$, and this is compatible with (6.35) if

$$\overline{\alpha} < \frac{1}{2\beta + C} \quad \text{or} \quad \overline{\alpha} < \frac{\delta\mu}{2(\mu C + 2C^*)},$$

respectively and depending on the minimum of (6.35).

Previous inequalities on the coefficients appearing in equation (6.28) guarantee that there exists a small coefficient $\iota > 0$ such that

$$\begin{aligned} \partial_t(\mathcal{E} + \overline{\overline{\alpha}}\mathcal{L}_1 + \overline{\overline{\alpha}}\mathcal{L}_2 + \overline{\alpha}(\partial_t \boldsymbol{v}, \boldsymbol{v}\rangle + \overline{\alpha}(\partial_t u, u\rangle) + \iota\mathcal{E} \\ \leq C\left(\|f_1\| + \|f_2\| + \|f_3\| + \|f_4\|\right)\mathcal{E}^{1/2}, \quad (6.36) \end{aligned}$$

where Schwartz inequality is applied in the right-hand side of (6.28).

We observe that the energy $\mathcal{E}$ and the functional

$$\mathcal{L} = \mathcal{E} + \overline{\overline{\alpha}}\mathcal{L}_1 + \overline{\overline{\alpha}}\mathcal{L}_2 + \overline{\alpha}\langle \partial_t \boldsymbol{v}, \boldsymbol{v}\rangle + \overline{\alpha}\langle \partial_t u, u\rangle$$

are equivalent when $\overline{\alpha}$ and $\overline{\overline{\alpha}}$ are chosen small enough. Indeed, using Schwartz inequality it is immediate to see that there exists $c_1 > 0$ such that

$$\mathcal{L} \leq c_1 \mathcal{E},$$

vice versa, when $\overline{\alpha}$ and $\overline{\overline{\alpha}}$ are small, $\mathcal{L}$ is a quadratic form positively defined, hence we find $c_2 > 0$ such that

$$\mathcal{E} \leq c_2 \mathcal{L}.$$

Then, we can rewrite equation (6.36) as

$$\partial_t \mathcal{L} + \epsilon\mathcal{L} \leq C\left(\|f_1\| + \|f_2\| + \|f_3\| + \|f_4\|\right)\mathcal{L}^{1/2},$$

for some $C, \epsilon > 0$. By virtue of a generalized Gronwall Lemma (see, e.g. Lemma 2.5 in [17]), we deduce

$$\begin{aligned} \mathcal{L}(t) \leq C_1 \mathcal{L}(0) e^{-\epsilon t} + \Big\{ C_2 \int_0^t e^{-\epsilon(t-\tau)/2} \left[\|f_1(\tau)\| + \|f_2(\tau)\|\right. \\ \left. + \|f_3(\tau)\| + \|f_4(\tau)\|\right] d\tau \Big\}^2. \end{aligned}$$

We deduce that a uniform energy estimate of this kind also holds for $\mathcal{E}$, regardless the names of the constants.

7. Proof of Corollary 5.2

Notice that the following inequality

$$\int_{\tau}^{t} m(y)e^{-\epsilon(t-y)}dy \leq \frac{e^{\epsilon}}{\epsilon}\sup_{r\geq\tau}\int_{r}^{r+1} m(y)dy$$

holds for any non-negative locally summable function m on $(\tau,+\infty)$. Hence,

$$C_2\int_{0}^{t} e^{-\epsilon/2(t-\tau)}\left[\|f_1(\tau)\| + \|f_2(\tau)\| + \|f_3(\tau)\| + \|f_4(\tau)\|\right]d\tau$$
$$\leq \frac{2e^{\epsilon/2}}{\epsilon}\sup_{r\geq 0}\int_{r}^{r+1}\left[\|f_1(\tau)\| + \|f_2(\tau)\| + \|f_3(\tau)\| + \|f_4(\tau)\|\right]d\tau = C_2^{1/2}(K).$$

This implies immediately the existence of a bounded absorbing set $\mathcal{B}_0$ in the phase space Z for the semigroup $S(t)$ associated to the solution of the problem: we can chose every ball $\mathcal{B}_0$ of Z centered at zero and of radius strictly greater than C_2.

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