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CONSTRUCTION OF CLASSICAL SOLUTIONS TO DIFFUSION EQUATIONS WITH VENTCEL'-VIŠIK BOUNDARY CONDITIONS

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Abstract: Ventcel's boundary condition is the most general lateral condition determining Markov extension of a minimal diffusion process governed by a diffusion operator on a smooth Euclidean or Riemannian domain. The boundary operator is given as a linear combination of a reflection operator, the boundary trace of the interior operator and a certain second order (possibly degenerate) elliptic type integro-differential operator. A Ventcel'-Višik boundary condition is local one without term of the boundary trace of the interior operator. Since the boundary operator has a term of second order differential operator, it is hard to apply a modern approach based on function spaces such as Sobolev space or weighted Hölder space to construct a weak solution or a fundamental solution. Using the classical method based on solving Volterra type integral equations with singular kernel, we construct a fundamental solution to diffusion equations with Ventcel'-Višik boundary condition under low regularity of coefficients.

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1. Introduction

To characterize Markov extension of a minimal diffusion process governed by a diffusion operator on a smooth Euclidean or Riemannian domain, Ventcel' [20] introduces a new class of boundary conditions. It includes classical boundary conditions such as Neumann and Dirichlet conditions and is so called Ventcel's boundary conditions. Each of the conditions is described by a boundary operator which is a linear combination of a reflection operator, a diffusion operator on the boundary, the boundary trace of the interior diffusion operator and a certain integral operator with order less than two.

Then the construction problem for a Markov process satisfying such a condition is investigated by Sato and Ueno [12], Bony, Courrège and Priouret [2], Cancelier [3] and Tira [14], [15] based on the semigroup theory in temporary homogeneous case. Garroni and Menaldi [5] give another approach based on sharp estimates on the fundamental solution to oblique derivative problem for a diffusion equation in the weighted Hölder space setting. More probabilistic approaches to the construction problem are done by several authors: Stroock and Varadhan [13] and Anderson [1] in the framework of the submartingale problem; Nakao [10], Nakao and Shiga [11], Watanabe [21] and Takanobu and Watanabe [16] in the framework of the theory of stochastic differential equations or of Poisson point processes.

Adapted to probability theory in temporary inhomogeneous case, we treat the terminal-boundary value problem instead of the initial-boundary value problem. From the viewpoint of PDE theory, the transition probability of a constructed Markov process can be regarded as a generalized solution to the terminal-boundary value problem for the diffusion equation with Ventcel's boundary condition. In the case with C^{∞} coefficients, Cattiaux [4] obtains a density of the transition probability by using the Malliavin calculus. However, under low regularity assumption on the coefficients, it is difficult to check that such a probabilistic solution becomes an ordinary function-type solution in strong or weak sense in the case with non-divergence main term or with second order boundary condition. Zeng and Luo [22] try to construct a classical solution in the case where the boundary operator is local (i.e. it has no the term of integral operator) and has a term of first order differential operator in time variable (this is equivalent to the existence of a term of the boundary trace of the interior diffusion operator) by showing the Schauder estimate. The former condition for the boundary operator means that the sample paths of the corresponding Markov process are continuous and the latter one means that the sample paths stay on the boundary with positive Lebesgue measure in time.

Hence under the latter condition, it is impossible to expect that the transition probability has a density with respect to the Lebesgue measure or Riemannian volume on the closure of the domain.

Now we focus on the case where the boundary operator is local and without a term of the boundary trace of the interior diffusion operator; we call such a boundary condition a Ventcel'-Višik boundary condition, following Bony-Courrège-Priouret [2]. Our purpose is to construct a fundamental solution to the terminal-boundary value problem for a diffusion equation with a non-divergence main term and with a Ventcel'-Višik boundary condition under assumptions on Hölder continuity of coefficients and on the non-degeneracy of the both diffusion and boundary operators. Since we need more effort to obtain the results on the explicit relationship between the fundamental solution and the corresponding diffusion process and on the regularity near the boundary of the fundamental solution (cf. [19]), we will discuss them elsewhere.

Here we mainly discuss the model problem (that is, the case where the domain is a half space in a Euclidean space); the treatment of a general case is fairly routine and tedious work (cf. [6], [7]). Hence we only give outline of the treatment in the general case. Following the oblique reflection case (cf. [18], [7]), we apply the parametrix method twice to construct a fundamental solution. Consider the case where the domain is the half space $\boldsymbol{R}^d_+ \equiv \boldsymbol{R}^{d-1} \times \boldsymbol{R}_+$ in $\boldsymbol{R}^d$. We take a fundamental solution to the model problem with frozen coefficients as a pre-parametrix. It can be given as the transition density of the solution to the corresponding stochastic differential equation. The explicit expression of the pre-parametrix is obtained through the skew product decomposition of the solution associated with the product decomposition of the domain (see [17]). It is represented as the sum of two terms, say the first and second terms; the first term denotes the minimal part corresponding to zero Dirichlet condition and the second term denotes the part associated with the boundary condition. In contrast to the case of oblique reflection, they have no common majorant: the first term has the usual Gaussian majorant but the second term does not. A majorant for the second term is obtained as a constant multiple of the fundamental solution to a model problem with constant coefficients in the case where each diffusion matrix of the interior and boundary operators is diagonal. Unfortunately, even if we use these majorants, we cannot immediately proceed in iteration; because the majorant for the second term has high singularity in time variable. Therefore, for the second term, we need to use a modified majorant together with the original one to proceed in iteration.

In Section 2, we treat the model problem. By providing the explicit form of the first parametrix and necessary estimates, we construct a desired fundamental solution. Section 3 is devoted to study the problem of constructing a fundamental solution in the general case.

2. The Model Problem

In this section, we take the domain $D = \boldsymbol{R}_+^d$ and construct a fundamental solution to the model problem on $\overline{D}$. The interior diffusion operator $\mathcal{A}$ and the boundary operator $\mathcal{B}$ are given in the following forms:

$$\mathcal{A}(s,x;\partial_s,\partial_x) := \frac{\partial}{\partial s} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(s,x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(s,x)\frac{\partial}{\partial x^i} + c(s,x)$$

$$(s \geq 0, x = (x^1,\ldots,x^d) \in \overline{D}),$$

$$\mathcal{B}(s,x;\partial_x) := \frac{\partial}{\partial x^d} + \frac{1}{2}\sum_{i,j=1}^{d-1} \alpha^{ij}(s,x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d-1} \beta^i(s,x)\frac{\partial}{\partial x^i} + \gamma(s,x)$$

$$(s \geq 0, x = (x^1,\ldots,x^d) \in \partial D).$$

For notational simplicity, we write

$$\mathcal{L} = 1_D\mathcal{A} + 1_{\partial D}\mathcal{B}.$$

Throughout this section, we assume that $\mathcal{A}$ and $\mathcal{B}$ are nondegenerate, that is, $(a^{ij}(s,x))$ and $(\alpha^{ij}(s,x))$ are uniformly elliptic, and that the coefficients of $\mathcal{A}$ and $\mathcal{B}$ are bounded and Hölder continuous with exponent $\lambda/2$ in s and λ in x for some $\lambda \in (0,1]$, say simply $(\lambda/2,\lambda)$–Hölder continuous in (s,x).

2.1. The Pre- and First Parametrix and their Estimates

To give the pre-parametrix, we consider the following operators $\mathcal{A}_0$ and $\mathcal{B}_0$ obtained from $\mathcal{A}$ and $\mathcal{B}$ by freezing the coefficients of the main terms as usual: for $\tau \geq 0$ and $\eta \in \overline{D}$, let

$$\mathcal{A}_0(\tau,\eta;\partial_s,\partial_x) := \frac{\partial}{\partial s} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(\tau,\eta)\frac{\partial^2}{\partial x^i \partial x^j},$$

$$\mathcal{B}_0(\tau,\eta;\partial_x) := \frac{\partial}{\partial x^d} + \frac{1}{2}\sum_{i,j=1}^{d-1} \alpha^{ij}(\tau,\eta)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d-1} \beta^i(\tau,\eta)\frac{\partial}{\partial x^i},$$

here, for $\eta = (\tilde{\eta}, \eta^d)$, we set $\alpha^{ij}(\tau, \eta) = \alpha^{ij}(\tau, \tilde{\eta}), \beta^i(\tau, \eta) = \beta^i(\tau, \tilde{\eta})$.

As the pre-parametrix, we take the fundamental solution $p^{\tau,\eta}(s, x; t, y)$ to the homogeneous terminal-boundary value problem $\mathcal{L}_0 = 1_D \mathcal{A}_0 + 1_{\partial D} \mathcal{B}_0 = 0$. To carry out calculation, we need the explicit expression of the fundamental solution. Now we note that the fundamental solution is the transition density of the diffusion process defined by the solution to the stochastic differential equation associated with $\mathcal{L}_0$. Then the explicit expression is obtained through a skew product decomposition of the solution process (see [17] in details). In order to express the explicit form of $p^{\tau,\eta}(s, x; t, y)$, we prepare some notations. Let $S \equiv (s^{ij}(\tau, \eta))$ be the symmetric positive square root of $(a^{ij}(\tau, \eta))$ and $s_i = s_i(\tau, \eta)$ the i-th column vector of S. Construct an orthonormal system $\{t_d, \ldots, t_1\}$ from $\{s_d, \ldots, s_1\}$ by the Gram-Schmidt orthogonalization and set $T := (t_1, \ldots, t_d)$, that is, the matrix consisting of the column vectors $t_1, \ldots, t_d$. Then $U := ST$ has the following decomposition:

$$U = \begin{bmatrix} \tilde{U} & \tilde{u}_d \\ 0 & u^{dd} \end{bmatrix},$$

where $\tilde{U} \equiv \tilde{U}(\tau, \eta)$ is a $(d-1) \times (d-1)$–matrix, $\tilde{u}_d \equiv \tilde{u}_d(\tau, \eta)$ is a $(d-1)$–column vector and $u^{dd} \equiv u^{dd}(\tau, \eta)$ is a positive number. Define two $(d-1) \times (d-1)$–matrices $\widetilde{V}$, $\widetilde{A}$ and a $(d-1)$–column vector $\tilde{\beta}$:

$$\widetilde{V} \equiv \widetilde{V}(\tau, \eta) := \widetilde{U}\widetilde{U}^*,$$

$$\widetilde{A} \equiv \widetilde{A}(\tau, \eta) := (\alpha^{ij}(\tau, \eta)),$$

$$\tilde{\beta} \equiv \tilde{\beta}(\tau, \eta) := (\beta^i(\tau, \eta)).$$

Let $g(t, u)$ be the one-dimensional standard Gauss kernel:

$$g(t, u) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) \quad (t > 0, u \in \boldsymbol{R}).$$

Define, for $t > 0, u \in \boldsymbol{R}$ and $\tilde{x} := (x^1, \ldots, x^{d-1}) \in \boldsymbol{R}^{d-1}$,

$$h(t, u) := -\frac{\partial g}{\partial u}(t, u), \quad G(t, \tilde{x}) := \prod_{i=1}^{d-1} g(t, x^i).$$

For $t > 0, \ell > 0$ and $x = (x^1, \ldots, x^d) = (\tilde{x}, x^d)$, $y = (y^1, \ldots, y^d) = (\tilde{y}, y^d) \in \overline{D}$, let

$$J(t, \ell; x, y)$$

$$
\begin{aligned}
&:= \int_{R^{d-1}} G\left(t-s, \tilde{U}^{-1}\left(\tilde{x}-\tilde{\zeta}+\frac{x^d-y^d-\ell}{u^{dd}}\tilde{u}_d+\ell\tilde{\beta}\right)\right)|\det \tilde{U}^{-1}| \\
&\qquad \times G(\ell, \tilde{S}^{-1}(\tilde{\zeta}-\tilde{y}))|\det \tilde{S}^{-1}|d\tilde{\zeta} \\
&= (2\pi)^{-(d-1)/2}\{\det(t\widetilde{V}+\ell\widetilde{A})\}^{-1/2}\exp\{-\frac{1}{2}(t\widetilde{V}+\ell\tilde{A}^{-1}\tilde{z}(\ell)\cdot\tilde{z}(\ell)\},
\end{aligned}
$$

where $\tilde{S}$ denotes the symmetric positive square root of $\tilde{A}$, $\tilde{z}(\ell)=(u^{dd})^{-1}(x^d-y^d-\ell)\tilde{u}_d+\ell\tilde{\beta}$ and the symbol $\cdot$ indicates the inner product in $\boldsymbol{R}^{d-1}$.

Then the first and second terms of the pre-parametrix, say $p_0^{\tau,\eta}(s,x;t,y)$ and $p_1^{\tau,\eta}(s,x;t,y)$, are expressed as follows: given $\tau\ge 0$ and $\eta\in\overline{D}$, let

$$
\begin{aligned}
p_0^{\tau,\eta}(s,x;t,y) &:= G\left(t-s, \tilde{U}^{-1}\left(\tilde{x}-\tilde{y}-\frac{x^d-y^d}{u^{dd}}\tilde{u}_d\right)\right) \\
&\qquad \times\left\{g\left(t-s,\frac{x^d-y^d}{u^{dd}}\right)-g\left(t-s,\frac{x^d+y^d}{u^{dd}}\right)\right\}\det S^{-1}, \quad (2.1)
\end{aligned}
$$

$$
p_1^{\tau,\eta}(s,x;t,y) := \frac{2}{u^{dd}}\int_0^\infty J(t-s,\varrho;x,y)h\left(t-s,\frac{x^d+y^d+\varrho}{u^{dd}}\right)d\varrho, \qquad (2.2)
$$

for $0\le s<t,\ x=(x^1,\dots,x^d)=(\tilde{x},x^d),\ y=(y^1,\dots,y^d)=(\tilde{y},y^d)\in\overline{D}$. Hence

$$
p^{\tau,\eta}(s,x;t,y)=p_0^{\tau,\eta}(s,x;t,y)+p_1^{\tau,\eta}(s,x;t,y)
$$

and it satisfies

$$
\int_{\overline{D}} p^{\tau,\eta}(s,x;t,y)dy=1.
$$

We first establish the following key inequality to get a majorant for the second term of the pre-parametrix.

Lemma 2.1. *Put $\tilde{w}:=\tilde{x}-\tilde{y}-(u^{dd})^{-1}(x^d-y^d)\tilde{u}_d$. For given $C_1, C_2, T>0$ and $\theta\in\boldsymbol{R}$, there exist positive constants $C_1'\in(0,C_1)$, $C_2'\in(0,C_1)$ and K such that*

$$
\begin{aligned}
&\int_0^\infty d\varrho(t-s+\varrho)^\theta\exp\left(-C_1\frac{(u^{dd})^{-2}\varrho^2}{t-s}\right)\exp\left(-C_2\frac{|\tilde{U}^{-1}\tilde{z}(\varrho)|^2}{t-s+\varrho}\right) \\
&\qquad \le K\int_0^\infty d\varrho(t-s+\varrho)^\theta\exp\left(-C_1'\frac{(u^{dd})^{-2}\varrho^2}{t-s}\right)\exp\left(-C_2'\frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right),
\end{aligned}
$$

for $0\le s<t\le T,\ x,y\in\overline{D},\ \tau\ge 0$ and $\eta\in\overline{D}$.

Proof. Let $\tilde{\delta} = -(u^{dd})^{-1}\tilde{u}_d + \tilde{\beta}$. Then $\tilde{z}(\varrho) = \tilde{w} + \varrho\tilde{\delta}$. Therefore

$$|\tilde{U}^{-1}\tilde{z}(\varrho)|^2 = |\tilde{U}^{-1}\tilde{w}|^2 + 2\rho\tilde{U}^{-1}\tilde{w}\cdot\tilde{U}^{-1}\tilde{\delta} + \varrho^2|\tilde{U}^{-1}\tilde{\delta}|^2.$$

Take $C_1' \in (0, C_1)$ and $C_2' \in (0, C_2)$ arbitrarily; then put

$$C_1'' := C_1 - C_1',\ C_2'' := C_2 - C_2'$$

and

$$T_0 := \frac{C_1''(C_2'')^2}{8(C_2)^3 M^3},$$

where $M := \sup_{\tau\geq 0, \eta\in\bar{D}} |\tilde{U}^{-1}\tilde{\delta}|$.

(i) We first verify the lemma in the case $0 < t - s \leq T_0$. The left hand side of the inequality is estimated as follows:

$$\begin{aligned}
&\int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1\frac{(u^{dd})^{-2}\varrho^2}{t-s}\right)\exp\left(-C_2\frac{|\tilde{U}^{-1}\tilde{z}(\varrho)|^2}{t-s+\varrho}\right)\\
&= \int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1\frac{(u^{dd})^{-2}\varrho^2}{t-s}\right)\exp\left(-C_2'\frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right)\\
&\qquad\times\exp\left(\frac{-C_2''|\tilde{U}^{-1}\tilde{w}|^2 - 2C_2\rho\tilde{U}^{-1}\tilde{w}\cdot\tilde{U}^{-1}\tilde{\delta} - C_2\varrho^2|\tilde{U}^{-1}\tilde{\delta}|^2}{t-s+\varrho}\right)\\
&\leq \int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1\frac{(u^{dd})^{-2}\varrho^2}{t-s}\right)\exp\left(-C_2'\frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right)\\
&\qquad\qquad\times\exp\left(\frac{-C_2''|\tilde{U}^{-1}\tilde{w}|^2 + 2C_2M\rho|\tilde{U}^{-1}\tilde{w}|}{t-s+\varrho}\right).
\end{aligned}$$

Then we divide the integral interval of the right hand side of the inequality above into three parts I_1, I_2, I_3:

$$\begin{aligned}
I_1 &:= \left(0, \sqrt{\frac{2C_2M}{C_1''}}|\tilde{U}^{-1}\tilde{w}|\sqrt{t-s}\right],\\
I_2 &:= \left(\sqrt{\frac{2C_2M}{C_1''}|\tilde{U}^{-1}\tilde{w}|\sqrt{t-s}}, \sqrt{\frac{2C_2M}{C_1''}|\tilde{U}^{-1}\tilde{w}|(t-s)}\right],\\
I_3 &:= \left(\sqrt{\frac{2C_2M}{C_1''}|\tilde{U}^{-1}\tilde{w}|(t-s)}, \infty\right),
\end{aligned}$$

where in the case $|\tilde{U}^{-1}\tilde{w}| \geq 1$ we regard as I_2 is empty. The integral on the interval I_1, say J_1, is estimated as

$$\begin{aligned} J_1 \leq &\int_{I_1} d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right) \\ &\times \exp\left(\frac{-C_2''|\tilde{U}^{-1}\tilde{w}|^2 + 2C_2 M\sqrt{(2C_2M)/C_1''}|\tilde{U}^{-1}\tilde{w}|^2\sqrt{t-s}}{t-s+\varrho}\right) \\ = &\int_{I_1} d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right) \\ &\times \exp\left(-\frac{|\tilde{U}^{-1}\tilde{w}|^2(C_2'' - 2C_2M\sqrt{(2C_2M/C_1'')(t-s)})}{t-s+\varrho}\right). \end{aligned}$$

Since $t-s \leq T_0$,

$$C_2'' - 2C_2M\sqrt{(2C_2M/C_1'')(t-s)} \geq 0.$$

This implies

$$J_1 \leq \int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right).$$

Next, in the case $|\tilde{U}^{-1}\tilde{w}| < 1$, we estimate the integral on the interval I_2, say J_2:

$$\begin{aligned} J_2 \leq &\int_{I_2} d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right) \\ &\times \exp\left(\frac{2C_2M\varrho|\tilde{U}^{-1}\tilde{w}|}{t-s+\varrho}\right) \\ \leq e^{2C_2M} &\int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right). \end{aligned}$$

Finally we estimate the integral on the interval I_3, say J_3:

$$\begin{aligned} J_3 \leq &\int_{I_3} d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right) \\ &\times \exp\left(2C_2M|\tilde{U}^{-1}\tilde{w}|\right) \end{aligned}$$

$$\leq \int_{I_3} d\varrho (t-s+\varrho)^\theta \exp\left(-C_1' \frac{(u^{dd})^{-2}\rho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right) \times \exp\left(-\frac{C_1''\varrho^2 - 2C_2 M|\tilde{U}^{-1}\tilde{w}|(t-s)}{t-s}\right)$$

$$\leq \int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1' \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right),$$

because $C_1''\varrho^2 - 2C_2 M|\tilde{U}^{-1}\tilde{w}|(t-s) \geq 0$ for $\varrho \in I_3$. Consequently we see that for $0 < t-s \leq T_0$

$$\int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2 \frac{|\tilde{U}^{-1}\tilde{z}(\varrho)|^2}{t-s+\varrho}\right)$$

$$\leq \left(e^{2C_2 M}+2\right) \int_0^\infty d\varrho (t-s+\rho)^\theta \exp\left(-C_1' \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right).$$

This proves the proposition in the case $0 < t-s \leq T_0$.

(ii) We regard the constant T_0 as a function of (C_1, C_1', C_2, C_2'), i.e., $T_0 = T_0(C_1, C_1', C_2, C_2')$. Then it is homogenous with degree zero in all arguments, that is, for $\kappa > 0$

$$T_0(\kappa C_1, \kappa C_1', \kappa C_2, \kappa C_2') = T_0(C_1, C_1', C_2, C_2').$$

For arbitrarily given $T > T_0$, it is enough to show the lemma. Let $\kappa := T_0/T$ and note that $(t-s+\varrho)^\theta \leq \kappa^{-|\theta|}(\kappa(t-s)+\varrho)^\theta$. Then it follows that

$$\int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2 \frac{|\tilde{U}^{-1}\tilde{z}(\varrho)|^2}{t-s+\varrho}\right) \leq$$

$$\kappa^{-|\theta|} \int_0^\infty d\varrho (\kappa(t-s)+\varrho)^\theta \exp\left(-\kappa C_1 \frac{(u^{dd})^{-2}\varrho^2}{\kappa(t-s)}\right) \exp\left(-\kappa C_2 \frac{|\tilde{U}^{-1}\tilde{w}|^2}{\kappa(t-s)+\varrho}\right).$$

Noting the inequality $0 < \kappa(t-s) \leq \kappa T = T_0$, we see that the right hand side of the inequality above is dominated by

$$\kappa^{-|\theta|}(e^{2\kappa C_2 M}+2) \int_0^\infty d\varrho (\kappa(t-s)+\varrho)^\theta \exp\left(-\kappa C_1' \frac{(u^{dd})^{-2}\varrho^2}{\kappa(t-s)}\right) \times \exp\left(-\kappa C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{\kappa(t-s)+\varrho}\right).$$

Since $(\kappa(t-s)+\varrho)^\theta \le \kappa^{-|\theta|}(t-s+\varrho)^\theta$, the quantity above is also dominated by

$$\kappa^{-2|\theta|}(e^{2\kappa C_2 M}+2) \times \int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1' \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-\kappa C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right).$$

Hence for $0 \le s < t \le T$

$$\int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1 \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \exp\left(-C_2 \frac{|\tilde{U}^{-1}\tilde{z}(\varrho)|^2}{t-s+\varrho}\right) \le \kappa^{-2|\theta|}(e^{2\kappa C_2 M}+2) \int_0^\infty d\varrho (t-s+\varrho)^\theta \exp\left(-C_1' \frac{(u^{dd})^{-2}\varrho^2}{t-s}\right) \times \exp\left(-\kappa C_2' \frac{|\tilde{U}^{-1}\tilde{w}|^2}{t-s+\varrho}\right).$$

This completes the proof of the lemma. □

Now we proceed to obtain basic estimates for the pre-parametrix. The first term $p_0^{\tau,\eta}(s,x;t,y)$ satisfies the usual Gaussian estimates, that is, there exist positive constants K and C such that

$$|\partial_s^m \partial_x^n p_0^{\tau,\eta}(s,x;t,y)| \le K(t-s)^{-(d+2m+n)/2} \exp\left(-C\frac{|x-y|^2}{t-s}\right)$$

for any $m, n \in \boldsymbol{Z}_+$, $0 \le s < t$, $x, y \in \overline{D}$, $\tau \ge 0$ and $\eta \in \overline{D}$. The second term $p_1^{\tau,\eta}(s,x;t,y)$ does not satisfy such Gaussian estimates. Hence, using Lemma 2.1, we show necessary estimates in the following. Since

$$|\tilde{U}^{-1}\tilde{w}|^2 + (u^{dd})^{-2}(x^d-y^d)^2 = |U^{-1}(x-y)|^2 \ge c|x-y|^2$$

with some positive constant c, we can ensure that for arbitrarily fixed $T > 0$ there exist positive constants K and C such that, for any $0 \le s < t \le T$, $x, y \in \overline{D}$, $\tau \ge 0$ and $\eta \in \overline{D}$,

$$p_1^{\tau,\eta}(s,x;t,y) \le K(t-s)^{-1} \exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right) \int_0^\infty d\varrho\ (t-s+\varrho)^{-(d-1)/2} \times \exp\left(-C\frac{\varrho^2}{t-s}\right) \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right),$$

$$
\begin{aligned}
&|\partial_{x^i}\partial_{x^j} p_1^{\tau,\eta}(s,x;t,y)| \\
&\quad \le K(t-s)^{-1}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)\int_0^\infty d\varrho\ (t-s+\varrho)^{-(d+1)/2} \\
&\qquad \times \exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right) \qquad (1\le i,j\le d-1),
\end{aligned}
$$

$$
\begin{aligned}
&|\partial_{x^i}\partial_{x^d} p_1^{\tau,\eta}(s,x;t,y)| \\
&\quad \le K(t-s)^{-3/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)\int_0^\infty d\varrho\ (t-s+\varrho)^{-d/2} \\
&\qquad \times \exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right) \qquad (1\le i\le d-1)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\partial_{x^d}^2 p_1^{\tau,\eta}(s,x;t,y)| \\
&\quad \le K(t-s)^{-2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)\int_0^\infty d\varrho\ (t-s+\varrho)^{-(d-1)/2} \\
&\qquad \times \exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right).
\end{aligned}
$$

Now we set

$$
\begin{aligned}
\bar{p}_i(s,x;t,y) &:= p_i^{t,y}(s,x;t,y) \qquad (i=0,1),\\
\bar{p}(s,x;t,y) &:= p^{t,y}(s,x;t,y).
\end{aligned}
$$

Here we take $\bar{p}(s,x;t,y)$ as the *first parametrix*. Furthermore we put

$$
\begin{aligned}
\bar{q}_i(s,x;t,y) &:= \mathcal{A}(s,x;\partial_s,\partial_x)\bar{p}_i(s,x;t,y) \qquad (i=0,1),\\
\bar{q}(s,x;t,y) &:= \mathcal{A}(s,x;\partial_s,\partial_x)\bar{p}(s,x;t,y).
\end{aligned}
$$

Then we can choose some positive constants K and C satisfying the inequalities:

$$
|\bar{q}_0(s,x;t,y)| \le K(t-s)^{-(d+2-\lambda)/2}\exp\left(-C\frac{|x-y|^2}{t-s}\right) \qquad (0\le s<t \text{ and } x,y\in\overline{D}), \quad (2.3)
$$

$$|\bar{q}_1(s,x;t,y)| \le K(t-s)^{-2+\lambda/4} \exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)$$
$$\times \int_0^\infty d\varrho\ (t-s+\varrho)^{-(d-1)/2} \exp\left(-C\frac{\varrho^2}{t-s}\right) \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right)$$
$$(0 \le s < t \le T \text{ and } x,y \in \overline{D}). \quad (2.4)$$

To carry out iteration, we must use another majorant for $\bar{q}_1(s,x;t,y)$; because the majorant given by the right hand side of (2.4) has high singularity in time variable. Hence, modifying it, we introduce another majorant for $\bar{q}_1(s,x;t,y)$ with low singularity in time variable. Take a positive integer m satisfying

$$1 - \frac{\lambda}{2} < \frac{m}{m+1},$$

and set

$$\mu := \frac{m}{m+1} \qquad \text{and} \qquad \nu := \frac{\lambda}{2} + \mu - 1.$$

Then, for $C' \in (0,C)$, replacing K by another constant K', we have

$$|\bar{q}_1(s,x;t,y)| \le K'(t-s)^{-3/2+\nu/2} \exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right) \times$$
$$\int_0^\infty d\varrho\ \varrho^{-\mu}\ (t-s+\varrho)^{-(d-1)/2} \exp\left(-C'\frac{\varrho^2}{t-s}\right) \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right), \quad (2.5)$$

for $0 \le s < t \le T$ and $x,y \in \bar{D}$. For the constants K, K' and C, C' in (2.3), (2.4) and (2.5), we can take them such as $K = K'$ and $C = C'$. That is, the estimate (2.5) is rewritten as

$$|\bar{q}_1(s,x;t,y)| \le K(t-s)^{-3/2+\nu/2} \exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right) \times$$
$$\int_0^\infty d\varrho\ \varrho^{-\mu}\ (t-s+\varrho)^{-(d-1)/2} \exp\left(-C\frac{\varrho^2}{t-s}\right) \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right), \quad (2.6)$$

for $0 \le s < t \le T$ and $x,y \in \overline{D}$. In the following, we suppose that the inequalities (2.3), (2.4) and (2.6) hold with the same constants K and C.

To simplify the notation in iteration, let us introduce the following functions Φ_0, Φ_1 and $\widetilde{\Phi}_1$ related to the majorants for $\bar{q}_0(s,t;x,y)$ and $\bar{q}_1(s,t;x,y)$. For $\kappa > 0$, $\theta < 1$ and $0 < \Xi \le C$, let

$$\Phi_0(s,x;t,y:\kappa,C) := (t-s)^{-(d+2-\kappa)/2} \exp\left(-C\frac{|x-y|^2}{t-s}\right),$$

$$\Phi_1(s,x;t,y:\kappa,C) := (t-s)^{-(4-\kappa)/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)$$
$$\times\int_0^\infty d\varrho\ (t-s+\varrho)^{-(d-1)/2}\exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right),$$
$$\widetilde{\Phi}_1(s,x;t,y:\kappa,\theta,\Xi,C) := (t-s)^{-(3-\kappa)/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)$$
$$\times\int_0^\infty d\varrho\ \varrho^{-\theta}\ (t-s+\varrho)^{-(d-1)/2}\exp\left(-\Xi\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right).$$

2.2. Construction of the Second Parametrix

We investigate the solvability of the integral equation:

$$\phi(s,x;t,y) = \bar{q}(s,x;t,y) + \int_s^t d\tau\int_D \bar{q}(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d\eta. \tag{2.7}$$

Fix $T>0$ and assume $0\le s<t\le T$ in this and next subsections. For two functions $f(s,x;t,y)$ and $g(s,x;t,y)$ ($0\le s<t\le T,\ x,y\in\overline{D}$), define the operation $*$ by

$$f*g(s,x;t,y) := \int_s^t d\tau\int_D f(s,x;\tau,\eta)g(\tau,\eta;t,y)d\eta.$$

Then the operation satisfies the associative law. If $f*g$ and $g*f$ have a common majorant Φ, that is, for any $0\le s<t\le T$ and $x,y\in\overline{D}$

$$|f*g(s,x,;t,y)|\le\Phi(s,x;t,y),\quad |g*f(s,x;t,y)|\le\Phi(s,x;t,y),$$

then we write it as

$$|f*g|\approx|g*f|\preceq\Phi.$$

The following inequalities are easily checked:

$$\Phi_0(\cdot:\kappa,C)*\Phi_0(\cdot:\kappa',C)\le\left(\frac{\pi}{C}\right)^{d/2}B\left(\frac{\kappa}{2},\frac{\kappa'}{2}\right)\Phi_0(\cdot:\kappa+\kappa',C); \tag{2.8}$$

$$\Phi_0(\cdot:\kappa,C)*\widetilde{\Phi}_1(\cdot:\kappa',\theta,\Xi,C)\approx\widetilde{\Phi}_1(\cdot:\kappa',\theta,\Xi,C)*\Phi_0(\cdot:\kappa,C)$$
$$\preceq\left(\frac{\pi}{C}\right)^{d/2}B\left(\frac{\kappa}{2},\frac{\kappa'}{2}\right)\widetilde{\Phi}_1(\cdot:\kappa+\kappa',\theta,\Xi,C); \tag{2.9}$$

$$\widetilde{\Phi}_1(\cdot : \kappa, \theta, \Xi, C) * \widetilde{\Phi}_1(\cdot : \kappa', \theta', \Xi', C) \le \left(\frac{\pi}{C}\right)^{d/2} B(\frac{\kappa}{2}, \frac{\kappa'}{2}) \times B(1-\theta, 1-\theta')\widetilde{\Phi}_1(\cdot : \kappa+\kappa', \theta+\theta'-1, \frac{\Xi\Xi'}{\Xi+\Xi'}, C); \quad (2.10)$$

$$\widetilde{\Phi}_1(\cdot : \kappa, 0, \Xi, C) * \Phi_1(\cdot : \kappa', C) \approx \Phi_1(\cdot : \kappa', C) * \widetilde{\Phi}_1(\cdot : \kappa, 0, \Xi, C) \preceq \left(\frac{\pi}{C}\right)^{d/2} \left(\frac{\pi}{\Xi}\right)^{1/2} \widetilde{\Phi}_1(\cdot : \kappa+\kappa', 0, \Xi, C); \quad (2.11)$$

$$\widetilde{\Phi}_1(\cdot : \kappa, 0, \Xi, C) * \Phi_1(\cdot : \kappa', C) * \Phi_0(\cdot : \kappa'', C) \approx \widetilde{\Phi}_1(\cdot : \kappa, 0, \Xi, C) * \Phi_0(\cdot : \kappa'', C) * \Phi_1(\cdot : \kappa', C) \preceq \left\{\left(\frac{\pi}{C}\right)^{d/2}\right\}^2 \left(\frac{\pi}{\Xi}\right)^{1/2} \frac{\Gamma(\frac{\kappa}{2})\Gamma(\frac{\kappa'}{2})\Gamma(\frac{\kappa''}{2})}{\Gamma(\frac{\kappa+\kappa'+\kappa''}{2})} \widetilde{\Phi}_1(\cdot : \kappa+\kappa'+\kappa'', 0, \Xi, C). \quad (2.12)$$

The solution $\phi(s, x; t, y)$ of the equation (2.7) is given by iteration.

Lemma 2.2. *Let*

$$\bar{q}_{(n)}(s, x; t, y) := \underbrace{\bar{q} * \cdots * \bar{q}}_{n}(s, x; t, y).$$

Then, for each $\delta > 0$, the series

$$\phi(s, x; t, y) = \sum_{n=1}^{\infty} \bar{q}_{(n)}(s, x; t, y)$$

converges absolutely and uniformly in s, t with $\delta \le t - s \le T$ and $x, y \in \overline{D}$. Moreover $\phi(s, x; t, y)$ satisfies the equation (2.7) and the following estimate with some positive constants K and C:

$$|\phi(s, x; t, y)| \le K(t-s)^{-(d+2-\lambda)/2} \exp\left(-C\frac{|x-y|^2}{t-s}\right) + K(t-s)^{-(3-\nu)/2} \exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right) \times \int_0^{\infty} d\varrho\ \varrho^{-\mu}(t-s+\varrho)^{-(d-1)/2} \exp\left(-C\frac{\varrho^2}{t-s}\right) \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right) \quad (2.13)$$

for $0 \le s < t \le T$ and $x, y \in \overline{D}$.

Proof. We have

$$
\begin{aligned}
|\bar q_{(n)}(s,x;t,y)| &= |(\bar q_0+\bar q_1)*\cdots*(\bar q_0+\bar q_1)(s,x;t,y)| \\
&\le \sum_{\varepsilon_1,\dots,\varepsilon_n=0,1} |\bar q_{\varepsilon_1}*\cdots*\bar q_{\varepsilon_n}(s,x;t,y)| \\
&= \sum_{j=0}^{m} \sum_{\substack{\varepsilon_1,\dots,\varepsilon_n=0,1\\ \varepsilon_1+\cdots+\varepsilon_n=j}} |\bar q_{\varepsilon_1}*\cdots*\bar q_{\varepsilon_n}(s,x;t,y)| \\
&+ \sum_{j=m+1}^{n} \sum_{\substack{\varepsilon_1,\dots,\varepsilon_n=0,1\\ \varepsilon_1+\cdots+\varepsilon_n=j}} |\bar q_{\varepsilon_1}*\cdots*\bar q_{\varepsilon_n}(s,x;t,y)| \\
&\approx \sum_{j=0}^{m} 2^n |\underbrace{\bar q_1*\cdots*\bar q_1}_{j}*\underbrace{\bar q_0*\cdots*\bar q_0}_{n-j}(s,x;t,y)| \\
&+ 2^n |\underbrace{\bar q_1*\cdots*\bar q_1}_{m+1}*\underbrace{\bar q_1*\cdots*\bar q_1}_{j-m-1}*\underbrace{\bar q_0*\cdots*\bar q_0}_{n-j}(s,x;t,y)|.
\end{aligned}
$$

In fact, we can verify the following inequalities. In the case $j=0$

$$
|\bar q_{\varepsilon_1}*\cdots*\bar q_{\varepsilon_n}| \le K^n \left\{ \left(\frac{\pi}{C}\right)^{d/2} \right\}^n \frac{\Gamma(\frac{\lambda}{2})^n}{\Gamma(\frac{n\lambda}{2})} \Phi_0(\cdot : n\lambda, C).
$$

In the case $1 \le j \le m$, using the majorants $K\Phi_0(\cdot : \lambda, C)$ for $\bar q_0$ and $K\widetilde\Phi_1(\cdot : \nu,\mu,C,C)$ for $\bar q_1$, we see that

$$
\begin{aligned}
|\bar q_{\varepsilon_1}*\cdots*\bar q_{\varepsilon_n}| &\approx |\underbrace{\bar q_1*\cdots*\bar q_1}_{j}*\underbrace{\bar q_0*\cdots*\bar q_0}_{n-j}| \\
&\preceq K^n \left\{ \left(\frac{\pi}{C}\right)^{d/2} \right\}^n \frac{\Gamma(1-\mu)}{\Gamma(j(1-\mu))} \frac{\Gamma(\frac{\nu}{2})^j \Gamma(\frac{\lambda}{2})^{n-j}}{\Gamma(\frac{j\nu+(n-j)\lambda}{2})} \\
&\qquad \times \widetilde\Phi_1(\cdot : j\nu+(n-j)\lambda, j\mu-(j-1), \frac{C}{j}, C).
\end{aligned}
$$

In the case $j \ge m+1$, we see that

$$
|\bar q_{\varepsilon_1}*\cdots*\bar q_{\varepsilon_n}| \approx |\underbrace{\bar q_1*\cdots*\bar q_1}_{m+1}*\underbrace{\bar q_1*\cdots*\bar q_1}_{j-(m+1)}*\underbrace{\bar q_0*\cdots*\bar q_0}_{n-j}|
$$

$$
\preceq K^{m+1} \left\{ \left(\frac{\pi}{C}\right)^{d/2} \right\}^m \frac{\Gamma(1-\mu)^{m+1}\Gamma(\frac{\nu}{2})^{m+1}}{\Gamma(\frac{(m+1)\nu}{2})} \widetilde\Phi_1(\cdot : (m+1)\nu, 0, \frac{C}{m+1}, C)
$$

$$*K^{j-m-1}\left\{\left(\frac{\pi}{C}\right)^{(d+1)/2}\right\}^{j-m-2}\frac{\Gamma(\frac{\lambda}{4})^{j-m-1}}{\Gamma(\frac{(j-m-1)\lambda}{4})}\Phi_1(\cdot:\frac{(j-m-1)\lambda}{2},C)$$

$$*K^{n-j}\left\{\left(\frac{\pi}{C}\right)^{(d+1)/2}\right\}^{j-m-2}\frac{\Gamma(\frac{\lambda}{2})^{n-j}}{\Gamma(\frac{(n-j)\lambda}{2})}\Phi_0(\cdot:(n-j)\lambda,C).$$

Here for $\bar{q}_1$ we used the majorant $K\widetilde{\Phi}_1(\cdot:\nu,\mu,C,C)$ in the first $m+1$ times iteration and the majorant $K\Phi_1(\cdot:\frac{\lambda}{2},C)$ in the next successive $j-m-1$ times iteration. Then

$$\begin{aligned}\|\bar{q}_{\varepsilon_1}*\cdots*\bar{q}_{\varepsilon_n}| \le K^n&\left\{\left(\frac{\pi}{C}\right)^{d/2}\right\}^{n-j+m+1}\left\{\left(\frac{\pi}{C}\right)^{(d+1)/2}\right\}^{j-m-2}\left(\frac{(m+1)\pi}{C}\right)^{1/2}\\ &\times\frac{\Gamma(1-\mu)^{m+1}\Gamma(\frac{\nu}{2})^{m+1}\Gamma(\frac{\lambda}{4})^{j-m-1}\Gamma(\frac{\lambda}{2})^{n-j}}{\Gamma(\frac{(m+1)\nu+(j-m-1)\lambda/2+(n-j)\lambda}{2})}\\ &\times\widetilde{\Phi}_1(\cdot:(m+1)\nu+(j-m-1)\lambda/2+(n-j)\lambda,0,\frac{C}{m+1},C);\end{aligned}$$

hence it holds

$$\begin{aligned}|\bar{q}_{\varepsilon_1}*\cdots*\bar{q}_{\varepsilon_n}| \le K^n&\left\{\left(\frac{\pi}{C}\right)^{d/2}\right\}^{n-j+m+1}\left\{\left(\frac{\pi}{C}\right)^{(d+1)/2}\right\}^{j-m-2}\left(\frac{(m+1)\pi}{C}\right)^{1/2}\\ &\times\frac{\Gamma(1-\mu)^{m+1}\Gamma(\frac{\nu}{2})^{m+1}\Gamma(\frac{\lambda}{4})^{n-m-1}}{\Gamma(\frac{(m+1)\nu+(n-m-1)\lambda/2}{2})}\\ &\times\widetilde{\Phi}_1(\cdot:(m+1)\nu+(j-m-1)\lambda/2+(n-j)\lambda,0,\frac{C}{m+1},C).\end{aligned}$$

Therefore the assertion on the convergence for the series $\sum_{n=1}^{\infty}\bar{q}_{(n)}(s,x;t,y)$ is established by the well-known property of the Gamma function, and $\phi(s,x;t,y)$ satisfies

$$\begin{aligned}|\phi(s,x;t,y)\le K_0(t-s)^{-(d+2-\lambda)/2}&\exp\left(-C\frac{|x-y|^2}{t-s}\right)\\ &+\sum_{j=1}^{m+1}K_j(t-s)^{-(3-j\nu)/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)\\ \times\int_0^{\infty}d\varrho\ \varrho^{-j\mu+(j-1)}(t-s+\varrho)^{-(d-1)/2}&\exp\left(-\frac{C}{j}\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right)\end{aligned}$$

with some positive constants K_j $(j = 0, 1, \dots, m+1)$. Hence if we take $K = \max_{j=0,1,\dots,m+1} K_j$ and if we replace C and $\frac{C}{j}$ $(j = 1, \dots, m+1)$ by $\frac{C}{m+1}$ and denote it again by C, then we have the conclusion of the lemma. □

Now we show the Hölder continuity of $\phi(s, x; t, y)$. Let

$$\Phi_0(\{s, s'\}, x; t, y : \kappa, C) := (t-s)^{-(d+2-\kappa)/2} \exp\left(-C\frac{|x-y|^2}{t-s'}\right),$$

$$\widetilde{\Phi}_1(\{s, s'\}, x; t, y : \kappa, \theta, C) := (t-s)^{-(3-\kappa)/2} \exp\left(-C\frac{(x^d+y^d)^2}{t-s'}\right)$$

$$\times \int_0^\infty d\varrho\ \varrho^{-\theta}\ (t-s+\varrho)^{-(d-1)/2} \exp\left(-C\frac{\varrho^2}{t-s'}\right) \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s'+\varrho}\right).$$

Then we can easily verify the followings propositions.

Proposition 2.3. *For $\lambda' \in (\nu, \lambda)$ and $\nu' \in [0, \nu)$, there exist positive constants K and C such that*

$$\begin{aligned}&|\phi(s, x; t, y) - \phi(s, x'; t, y)| \\ &\quad \le K|x-x'|^{\lambda'} \Big\{\Phi_0(s, x; t, y : \lambda-\lambda', C) + \Phi_0(s, x'; t, y : \lambda-\lambda', C)\Big\} \\ &+ K|x-x'|^{\nu'} \Big\{\widetilde{\Phi}_1(s, x; t, y : \nu-\nu', \mu, C, C) + \widetilde{\Phi}_1(s, x'; t, y : \nu-\nu', \mu, C, C)\Big\}\end{aligned}$$

for $0 \le s < t \le T$, $x, x' \in \bar{D}$ and $y \in \bar{D}$;

$$\begin{aligned}|\phi(s, x; t, y) - \phi(s', x; t, y)| &\le K|s-s'|^{\lambda'/2}\Phi_0(\{s, s'\}, x; t, y : \lambda-\lambda', C) \\ &\quad + K|s-s'|^{\nu'/2}\widetilde{\Phi}_1(\{s, s'\}, x; t, y : \nu-\nu', \mu, C, C)\end{aligned}$$

for $0 \le s' \le s < t \le T$ and $x, y \in \overline{D}$.

Proposition 2.4. *For $T > 0$ and positive integers m, n, there exit positive constants C, K such that*

$$\begin{aligned}&|\partial_s^m \partial_x^n p_0^{\tau,\xi}(s, x; t, y) - \partial_s^m \partial_x^n p_0^{\tau,\eta}(s, x; t, y)| \\ &\qquad \le K|\xi-\eta|^\lambda (t-s)^{-(d+2m+n)/2} \exp\left(-C\frac{|x-y|^2}{t-s}\right)\end{aligned}$$

and

$$|\partial_s^m \partial_x^n p_1^{\tau,\xi}(s, x; t, y) - \partial_s^m \partial_x^n p_1^{\tau,\eta}(s, x; t, y)|$$

$$\le K|\xi-\eta|^{\lambda}(t-s)^{-(2+2m+n)/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)$$
$$\times\int_0^\infty d\varrho\ (t-s+\varrho)^{-(d-1)/2}\times\exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right)$$

for $0\le s<t\le T$ and $x,y\in\overline{D}$. In particular, for $1\le i,j\le d-1$

$$|\frac{\partial^2 p_1^{\tau,\xi}}{\partial x^i\partial x^j}(s,x;t,y)-\frac{\partial^2 p_1^{\tau,\eta}}{\partial x^i\partial x^j}(s,x;t,y)|$$
$$\le K|\xi-\eta|^{\lambda}(t-s)^{-1}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)$$
$$\times\int_0^\infty d\varrho\ (t-s+\varrho)^{-(d+1)/2}\times\exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right).$$

These results imply the following propositions as in the case without boundary condition (cf. [9]) or of oblique reflection (cf. [18]) .

Proposition 2.5. *If $2m+n\ge 1$ then*

$$\left|\partial_s^m\partial_x^n\int_D\bar{p}(s,x;t,\eta)d\eta\right|\le K(t-s)^{-(2m+n-\lambda)/2}.$$

Let

$$\bar{p}'(s,x;t,y):=\int_s^t d\tau\int_D\bar{p}(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d\eta.$$

Proposition 2.6. *(i) For $0\le s<t\le T$ and $x,y\in\bar{D}$*

$$\partial_s\bar{p}'(s,x;t,y)=\int_{(s+t)/2}^t d\tau\int_D\partial_s\bar{p}(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d\eta$$
$$+\int_s^{(s+t)/2}d\tau\int_D\partial_s\bar{p}(s,x;\tau,\eta)\{\phi(\tau,\eta;t,y)-\phi(\tau,x;t,y)\}d\eta$$
$$+\int_s^{(s+t)/2}\phi(\tau,x;t,y)d\tau\int_D\partial_s\bar{p}(s,x;\tau,\eta)d\eta-\phi(s,x;t,y),$$

and $\partial_s\bar{p}'(s,x;t,y)$ is continuous in $(s,x)\in[0,t)\times\overline{D}$ for $t>0,y\in\overline{D}$.

(ii) For $0\le s<t\le T$ and $x,y\in\overline{D}$

$$\partial_x^2\bar{p}'(s,x;t,y)=\int_{(s+t)/2}^t d\tau\int_D\partial_x^2\bar{p}(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d\eta$$

$$+\int_s^{(s+t)/2} d\tau \int_D \partial_x^2 \bar{p}(s,x;\tau,\eta)\{\phi(\tau,\eta;t,y)-\phi(\tau,x;t,y)\}d\eta$$
$$+\int_s^{(s+t)/2} \phi(\tau,x;t,y)d\tau \int_D \partial_x^2 \bar{p}(s,x;\tau,\eta)d\eta,$$

and $\partial_x^2 \bar{p}'(s,x;t,y)$ *is continuous in* $(s,x)\in[0,t)\times\overline{D}$ *for* $t>0, y\in\overline{D}$.

Proposition 2.7. *If* $2m+n\le 2$, *then for some* $\lambda''\in(0,\lambda)$ *and* $\nu''\in(0,\nu)$

$$|\partial_s^m \partial_x^n \bar{p}'(s,x;t,y)| \le K(t-s)^{-(d+2m+n-\lambda'')/2}\exp\left(-C\frac{|x-y|^2}{t-s}\right)$$
$$+K(t-s)^{-(1+2m+n-\nu'')/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)$$
$$\times\int_0^\infty d\varrho\varrho^{-\mu}(t-s+\varrho)^{-(d-1)/2}\exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right)$$

for $0\le s<t\le T$ *and* $x,y\in\overline{D}$.

Finally we define the *second parametrix* $\tilde{p}(s,x;t,y)$ by

$$\begin{aligned}\tilde{p}(s,x;t,y) &:= \bar{p}(s,x;t,y)+\int_s^t d\tau\int_D \bar{p}(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d\eta\\ &\equiv \bar{p}(s,x;t,y)+\bar{p}'(s,x;t,y).\end{aligned}$$

It satisfies

$$\tilde{p}(s,x;t,y)\in C^{1,2}([0,t)\times\overline{D}),$$
$$\mathcal{A}\tilde{p}(s,x;t,y)=0 \quad \text{for } (s,x)\in[0,t)\times D,$$

and the following equality also holds:

$$\mathcal{B}(s,x;\partial_x)\bar{p}'(s,x;t,y)=\int_s^t d\tau\int_D \mathcal{B}(s,x;\partial_x)\bar{p}(s,x;\tau,\eta)\phi(\tau,\eta;t,y).$$

Moreover if $2m+n\le 2$, then

$$|\partial_s^m \partial_x^n \tilde{p}(s,x;t,y)| \le K(t-s)^{-(d+2m+n)/2}\exp\left(-C\frac{|x-y|^2}{t-s}\right)$$
$$+K(t-s)^{-(2+2m+n-\mu)/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)$$
$$\times\int_0^\infty d\varrho\varrho^{-\mu}(t-s+\varrho)^{-(d-1)/2}\exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right)$$

for $0\le s<t\le T$ and $x,y\in\overline{D}$.

2.3. Construction of the Fundamental Solution

For $x \in \partial D, y \in \overline{D}$ and $0 \le s < t \le T$, we set

$$\tilde{q}(s,x;t,y) := \mathcal{B}(s,x;\partial_x)\tilde{p}(s,x;t,y).$$

We next investigate the solvability of the integral equation:

$$\psi(s,x;t,y) = \tilde{q}(s,x;t,y) + \int_s^t d\tau \int_{\partial D} \tilde{q}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)\sigma(d\eta), \qquad (2.14)$$

where $\sigma(d\eta)$ is the surface measure on ∂D, i.e., $\sigma(d\eta) = d\tilde{\eta}\delta_0(d\eta^d)$.

Let

$$\Psi(s,x;t,y:\kappa,\theta,\Xi,C) := (t-s)^{-1+\kappa/2} \exp\left(-C\frac{(y^d)^2}{t-s}\right)$$
$$\times \int_0^\infty d\varrho\ \varrho^{-\theta}\ (t-s+\varrho)^{-(d-1)/2} \exp\left(-\Xi\frac{\varrho^2}{t-s}\right) \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right).$$

Then

$$|\tilde{q}(s,x;t,y)| \le K\Psi(s,x;t,y:\nu,\mu,C,C)$$

with some positive constants K and C. For two functions $f(s,x;t,y)$ and $g(s,x;t,y)$ $(0 \le s < t \le T,\ x \in \partial D, y \in \overline{D})$, define the operation $\tilde{*}$ by

$$f\tilde{*}g(s,x;t,y) := \int_s^t d\tau \int_{\partial D} f(s,x;\tau,\eta)g(\tau,\eta;t,y)\sigma(d\eta).$$

This operation also satisfies the associative law. Let

$$\tilde{q}_{(n)}(s,x;t,y) := \underbrace{\tilde{q}\tilde{*}\cdots\tilde{*}\tilde{q}}_{n}(s,x;t,y).$$

As before, we have the following estimates: for $j = 1,2,\ldots,m+1$

$$|\tilde{q}_{(j)}(s,x;t,y)|$$
$$\le K^j \left\{\left(\frac{\pi}{C}\right)^{(d-1)/2}\right\}^{j-1} \frac{\Gamma(\frac{\nu}{2})^j}{\Gamma(\frac{j\nu}{2})} \Psi(s,x;t,y:j\nu,j\mu-j+1,\frac{C}{j},C);$$

and, for $\ell = 1,2,\ldots,$

$$|\tilde{q}_{(\ell(m+1))}(s,x;t,y)| \le K^{\ell(m+1)} \left\{\left(\frac{\pi}{C}\right)^{(d-1)/2}\right\}^{\ell(m+1)-1} \frac{\Gamma(\frac{\nu}{2})^{\ell(m+1)}}{\Gamma(\frac{\ell(m+1)\nu}{2})}$$

$$\times\left\{\left(T\frac{(m+1)\pi}{C}\right)^{1/2}\right\}^{\ell-1}\Psi(s,x;t,y:\ell(m+1)\nu,0,\frac{C}{m+1},C).$$

This implies, for $n=\ell(m+1)+j \quad (j=0,1,\ldots,m;\ \ell=1,2,\ldots)$

$$|\tilde{q}_{(n)}(s,x;t,y)|=|\tilde{q}_{(\ell(m+1))}\tilde{*}\tilde{q}_{(j)}(s,x;t,y)|$$
$$\leq K^n\left\{\left(\frac{\pi}{C}\right)^{(d-1)/2}\right\}^{n-1}\frac{\Gamma(\frac{\nu}{2})^n}{\Gamma(\frac{n\nu}{2})}\left(T\frac{(m+1)\pi}{C}\right)^{n/2(m+1)}(1-\mu)^{-1}$$
$$\times\Psi(s,x;t,y:n\nu,j(\mu-1),\frac{C}{m+1+j},C).$$

Hence it follows that for some constant L

$$|\tilde{q}_{(n)}(s,x;t,y)|\leq LK^n\left\{\left(\frac{\pi}{C}\right)^{(d-1)/2}\right\}^{n-1}\frac{\Gamma(\frac{\nu}{2})^n}{\Gamma(\frac{n\nu}{2})}\left(T\frac{(m+1)\pi}{C}\right)^{n/2(m+1)}$$
$$\times\Psi(s,x;t,y:n\nu,0,\frac{C}{2(m+1)},C).$$

Consequently we have the following lemma.

Lemma 2.8. *For each $\delta>0$, the series*

$$\psi(s,x;t,y)=\sum_{n=1}^{\infty}\tilde{q}_{(n)}(s,x;t,y)$$

converges absolutely and uniformly in s,t with $\delta\leq t-s\leq T$ and $x\in\partial D, y\in\overline{D}$. Moreover $\psi(s,x;t,y)$ satisfies the equation (2.14) and the following estimates with some positive constants K and C:

$$(i)\quad |\psi(s,x;t,y)|\leq K(t-s)^{-1+\nu/2}\exp\left(-C\frac{(y^d)^2}{t-s}\right)$$
$$\times\int_0^{\infty}d\varrho\ \varrho^{-\mu}(t-s+\varrho)^{-(d-1)/2}\exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right)\quad (2.15)$$

for $0\leq s<t\leq T$ and $x\in\partial D, y\in\overline{D}$.

(ii) for each $\lambda'\in(0,\lambda)$, we can choose $\mu',\nu'\in(0,1)$ so that

$$|\psi(s,x;t,y)-\psi(s,x';t,y)|\leq K|x-x'|^{\lambda'}(t-s)^{-1+\nu'/2}\exp\left(-C\frac{(y^d)^2}{t-s}\right)$$

$$
\times \int_0^\infty d\varrho\ \varrho^{-\mu'}(t-s+\varrho)^{-(d-1)/2} \exp\left(-C\frac{\varrho^2}{t-s}\right) \times \left\{ \exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right) + \exp\left(-C\frac{|\tilde{x'}-\tilde{y}|^2}{t-s+\varrho}\right)\right\} \quad (2.16)
$$

for $0 \le s < t \le T$, $x, x' \in \partial D$ *and* $y \in \overline{D}$.

Moreover we have the next results in an analogous way to the case of oblique reflection (cf. [18]); hence their proofs are omitted, although some of the proofs are more complicate.

Proposition 2.9. *(i) If* $x \in D$ *and* $2m + n \le 2$, *then*

$$
\partial_s^m \partial_x^n \int_s^t d\tau \int_{\partial D} \tilde{p}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)\sigma(d\eta) = \int_s^t d\tau \int_{\partial D} \partial_s^m \partial_x^n \tilde{p}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)\sigma(d\eta),
$$

and the above is continuous in $(s,x) \in [0,t) \times D$ *for* $t > 0, y \in \overline{D}$.

(ii) Let $x_o \in \partial D$ *and* $1 \le i,j \le d-1$

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^i \partial x^j} \int_s^t d\tau \int_{\partial D} \tilde{p}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)\sigma(d\eta)\ \Big|_{x=x_o} \\
&= \int_s^t d\tau \int_{\partial D} \frac{\partial^2 \tilde{p}}{\partial x^i \partial x^j}(s,x;\tau,\eta)\{\psi(\tau,\eta;t,y) - \psi(\tau,x_o;t,y)\}\sigma(d\eta) \\
&\qquad + \int_s^t \psi(\tau,x_o;t,y)d\tau \int_{\partial D} \frac{\partial^2 \tilde{p}}{\partial x^i \partial x^j}(s,x_o;\tau,\eta)\sigma(d\eta).
\end{aligned}
$$

(iii) For $x_o \in \partial D$

$$
\begin{aligned}
&\frac{\partial}{\partial x^d} \int_s^t d\tau \int_{\partial D} \tilde{p}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)\sigma(d\eta)\ \Big|_{x=x_o} \\
&= \int_s^t d\tau \int_{\partial D} \frac{\partial \tilde{p}}{\partial x^d}(s,x;\tau,\eta)\{\psi(\tau,\eta;t,y) - \psi(\tau,x_o;t,y)\}\sigma(d\eta) \\
&+ \int_s^t \psi(\tau,x_o;t,y)d\tau \int_{\partial D} \frac{\partial \tilde{p}}{\partial x^d}(s,x_o;\tau,\eta)\sigma(d\eta) - \psi(s,x_o;t,y).
\end{aligned}
$$

Proposition 2.10. *If* $x_o \in \partial D$, *then*

$$
\begin{aligned}
&\mathcal{B}(s,x;\partial x) \int_s^t d\tau \int_{\partial D} \tilde{p}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)\sigma(d\eta)\ \Big|_{x=x_o} \\
&= \int_s^t d\tau \int_{\partial D} \mathcal{B}(s,x;\partial x)\tilde{p}(s,x;\tau,\eta)\ \Big|_{x=x_o} \psi(\tau,\eta;t,y)\sigma(d\eta) - \psi(s,x_o;t,y)\,.
\end{aligned}
$$

In what follows, denote by $C(\overline{D})$ the space of continuous functions on $\overline{D}$, and by $C^{m,n}([0,t)\times E)$ the space of functions of $(s,x)\in[0,t)\times E$ ($E\subset\overline{D}$) such that those derivatives in s up to order m and those derivatives in x up to order n are continuous in (s,x) respectively. Moreover, $C^{0,2*}([0,t)\times\overline{D})$ indicates the space of functions $f(s,x)$ on $[0,t)\times\overline{D}$ for which $f\in C^{0,1}([0,t)\times\overline{D})$ and $\dfrac{\partial^2 f}{\partial x_i\partial x_j}(s,x)\quad(1\le i,j\le d-1)$ are continuous on $[0,t)\times\partial D$.

Let

$$p(s,x;t,y):=\tilde{p}(s,x;t,y)+\int_s^t d\tau\int_{\partial D}\tilde{p}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)\sigma(d\eta).$$

Then we see that for each $t>0$ and $y\in\overline{D}$

$$p(\cdot,\cdot;t,y)\in C^{1,2}([0,t)\times D)\cap C^{0,2*}([0,t)\times\overline{D}),$$

$$\mathcal{A}(s,x;\partial_s,\partial_x)p(s,x;t,y)=0\quad\text{for}\quad(s,x)\in[0,t)\times D,$$

$$\mathcal{B}(s,x;\partial_x)p(s,x;t,y)=0\quad\text{for}\quad(s,x)\in[0,t)\times\partial D,$$

and that

$$\lim_{s\uparrow t}\int_{\overline{D}}p(s,x;t,\eta)f(\eta)d\eta=f(x)$$

for any $x\in\overline{D}$ and bounded function f on $\overline{D}$. Therefore $p(s,x;t,y)$ becomes a *fundamental solution* to the terminal-value problem for $\mathcal{L}u=0$. Moreover it satisfies the following eatimate:

$$\begin{aligned}0\le p(s,x;t,y)\le{}& K(t-s)^{-d/2}\exp\left(-C\frac{|x-y|^2}{t-s}\right)\\&+K(t-s)^{-1+\mu/2}\exp\left(-C\frac{(x^d+y^d)^2}{t-s}\right)\\\times\int_0^\infty d\varrho\varrho^{-\mu}(t-s+\varrho)^{-(d-1)/2}&\exp\left(-C\frac{\varrho^2}{t-s}\right)\exp\left(-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s+\varrho}\right)\end{aligned}$$

for $0\le s<t\le T$ and $x,y\in\overline{D}$; here its nonnegativity is derived from the maximum principle.

3. The General Case

This section is devoted to the general case. In the rest of the paper, M denotes a d–dimensional $C^{2,\lambda}$–Riemannian manifold with a Riemannian metric g, and D is a relatively compact $C^{2,\lambda}$–domain of M, that is, ∂D is a $C^{2,\lambda}$–submanifold in M. Let $\mathcal{O} := \{(O_\kappa, \varphi_\kappa)\}_{\kappa\in\Lambda}$ be an atlas on M. Then we use the following notation:

$$\Lambda_\circ := \{\kappa \in \Lambda : O_\kappa \subset D\}, \qquad \Lambda_\partial := \{\kappa \in \Lambda : O_\kappa \cap \partial D \neq \phi\},$$
$$U_\kappa := O_\kappa \cap D, \quad V_\kappa := O_\kappa \cap \partial D, \quad W_\kappa := O_\kappa \cap \overline{D}.$$

Denote $\varphi_\kappa(x)$ by $x_\kappa = (x_\kappa^1, \dots, x_\kappa^d)$ and the Jacobi matrix of the map $x_\rho = \varphi_\rho \circ \varphi_\kappa^{-1}(x_\kappa)$ by $\partial x_\rho/\partial x_\kappa(x)$. For simplicity, we write x for x_κ as above when no confusion occurs.

We take a diffusion operator $\mathcal{A} \equiv \mathcal{A}(s, x; \partial_s, \partial_x)$ acts on functions defined on $[0,\infty) \times D$ and a boundary operator $\mathcal{B} \equiv \mathcal{B}(s, x; , \partial_x)$ which defines a Ventcel'-Višik condition:

$$\mathcal{A}(s, x; \partial_s, \partial_x) = \frac{\partial}{\partial s} + \tilde{\mathcal{A}}(s, x; \partial_x),$$

where $\tilde{\mathcal{A}}(s, x; \partial_x)$ is an elliptic operator on D with time parameter s, and

$$\mathcal{B}(s, x; \partial_x) = \upsilon(s, x) + \tilde{\mathcal{B}}(s, x; \partial_x),$$

where $\upsilon(s, x)$ is a vector field along ∂D being toward the inside and $\tilde{\mathcal{B}}(s, x; \partial_x)$ is an elliptic operator on ∂D with time parameter s; we often write $\upsilon(s, x)$ as $\dfrac{\partial}{\partial \upsilon(s, x)}$. They have the following forms locally:

$$\tilde{\mathcal{A}}_\kappa(s, x; \partial_x) := \frac{1}{2}\sum_{i,j=1}^{d} a_\kappa^{ij}(s, x)\frac{\partial^2}{\partial x_\kappa^i \partial x_\kappa^j} + \sum_{i=1}^{d} b_\kappa^i(s, x)\frac{\partial}{\partial x_\kappa^i} + c(s, x)$$

for $(s, x) \in [0,\infty) \times W_\kappa$, and

$$\frac{\partial}{\partial \upsilon(s, x)} := \sum_{i=1}^{d} \upsilon_\kappa^i(s, x)\frac{\partial}{\partial x_\kappa^i},$$

$$\tilde{\mathcal{B}}_\kappa(s, x; \partial_x) := \frac{1}{2}\sum_{i,j=1}^{d-1} \alpha_\kappa^{ij}(s, x)\frac{\partial^2}{\partial x_\kappa^i \partial x_\kappa^j} + \sum_{i=1}^{d-1} \beta_\kappa^i(s, x)\frac{\partial}{\partial x_\kappa^i} + \gamma(s, x)$$

for $(s, x) \in [0,\infty) \times V_\kappa$.

Then the coefficients of these operators satisfy suitable transformation rules respectively. In particular, for each $s \in [0, \infty)$, $(a_\kappa^{ij}(s,x))$ and $(\alpha_\kappa^{ij}(s,x))$ define tensor fields $a(s,x)$ on $\overline{D}$ and $\alpha(s,x)$ on ∂D, respectively. Later we use the concept of the Hölder continuity for the quantities (such as scalar, vector, tensor fields, etc.) which determine the diffusion and boundary operators. For a detailed arguments for the Hölder continuity of such quantities, we refer to [7].

3.1. Assumption and the Result

As before, to treat the diffusion and boundary operators simultaneously, we introduce the operator $\mathcal{L} \equiv \mathcal{L}(s,x;\partial_s,\partial_x)$ acting on functions defined on $[0,\infty)\times \overline{D}$ by

$$\mathcal{L}(s,x;\partial_s,\partial_x) := 1_D(x)\mathcal{A}(s,x;\partial_s,\partial_x) + 1_{\partial D}(x)\mathcal{B}(s,x;\partial_x),$$

where $1_A(x)$ denotes the indicator of the set A. That is, the form of $\mathcal{L}$ on each chart O_κ is given by

$$\mathcal{L}_\kappa(s,x;\partial_s,\partial_x) := 1_{U_\kappa}(x)\mathcal{A}_\kappa(s,x;\partial_s,\partial_x) + 1_{V_\kappa}(x)\mathcal{B}_\kappa(s,x;\partial_x)$$

for $(s,x) \in [0,\infty) \times W_\kappa$.

For a smooth function f on M, $\nabla_x f(x)$ and $\nabla_x^2 f(x)$ stand for the gradient and the Hessian of f at x, respectively, with respect to the Levi-Civita connection induced by the Riemannian metric g. Their norms are defined through an orthonormal basis $(e_1,\dots,e_d)$ of the tangent space T_xM with respect to g:

$$|\nabla_x f(x)|^2 := \sum_{i=1}^{d} |(\nabla_x)_{e_i} f(x)|^2, \qquad |\nabla_x^2 f(x)|^2 := \sum_{i,j=1}^{d} |(\nabla_x^2)_{e_i,e_j} f(x)|^2,$$

which are independent of the choice of $(e_1,\dots,e_d)$. For the Riemannian metric g, denote by $d_g(\,\cdot\,,\,\cdot\,)$ and by $v_g(dy)$ the associated distance function and Riemannian volume, respectively.

Here we use the term of a fundamental solution in the following sense.

Definition 3.1. A function $p(s,x;t,y)$ $(0 \le s < t;\ x,y \in \overline{D})$ is called a *fundamental solution* to the terminal value problem for $\mathcal{L}u = 0$ if $p(s,x;t,y)$ satisfies the following conditions:

1. $p(s,x;t,y)$ is continuous in (s,x,t,y).
2. For any $t > 0$ and $y \in \overline{D}$,

$$p(\cdot,\cdot;t,y) \in C^{1,2}([0,t)\times D) \cap C^{0,2*}([0,t)\times \overline{D})$$

and

$$\mathcal{L}(s,x;\partial_s,\partial_x)p(s,x;t,y)=0$$

for $(s,x)\in[0,t)\times\overline{D}$.

3. For each $t>0$ and $\varepsilon>0$,

$$\sup_{(s,x)\in[0,t)\times\overline{D}}|p(s,x;t,y)|,\quad \sup_{(s,x)\in[0,t-\varepsilon)\times\overline{D}}|\nabla_x p(s,x;t,y)|,$$
$$\sup_{(s,x)\in[0,t-\varepsilon)\times\overline{D_{g,\varepsilon}}}|\partial_s^m\nabla_x^n p(s,x;t,y)|$$

are integrable with respect to $v_g(dy)$, where $D_{g,\varepsilon}:=\{x\in D : d_g(x,\partial D)>\varepsilon\}$ and $(m,n)=(1,0)$ or $(0,2)$.

4. For any $f\in C(\overline{D})$,

$$\lim_{s\uparrow t}\int_{\overline{D}}p(s,x;t,y)f(y)v_g(dy)=f(x).$$

Now we state a condition for the coefficients of $\mathcal{L}$.

(L–1)
1. The tensor field $a(s,x)$ on $\overline{D}$ is symmetric and positive definite.
2. The quantities which determine the diffusion operator $\mathcal{A}$ are $(\lambda/2,\lambda)$–Hölder continuous in (s,x) of $[0,\infty)\times\overline{D}$.
3. The vector field $\upsilon(s,x)$ is toward the inside, that is,

$$g\left(\upsilon(s,x),n_g(x)\right)>0$$

for any $(s,x)\in[0,\infty)\times\partial D$, where $n_g(x)$ denotes the unit inward normal vector field along ∂D with respect to the metric g.

4. The tensor field $\alpha(s,x)$ on ∂D is symmetric and positive definite.
5. The quantities which determine the boundary operator $\mathcal{B}$ are $(\lambda/2,\lambda)$–Hölder continuous in (s,x) of $[0,\infty)\times\partial D$.

Then we have the following result.

Theorem 3.2. *Assume the condition (L–1). Then there exists a fundamental solution $p(s,x;t,y)$ to the terminal-value problem for $\mathcal{L}u=0$. Moreover, for bounded continuous function f on $\overline{D}$ and $(s,x)\in[0,t]\times\overline{D}$, define*

$$u(s,x)=\int_{\overline{D}}p(s,x;t,\eta)f(\eta)v_g(d\eta).$$

Then $u(s,x)\in C^{1,2}([0,t)\times D)\cap C^{0,2*}([0,t)\times\overline{D})$ is a solution to the equation $\mathcal{L}(s,x;\partial_s,\partial_x)u(s,x)=0$ with terminal value $f(x)$.

3.2. Outline of the Construction of the Fundamental Solution

The framework of the construction is almost the same as in the case of oblique reflection. Hence we describe its outline following [7]. The coefficients of the operator $\mathcal{L}$ satisfy the transformation rule; hence it suffices to carry out the construction through an appropriately fixed atlas on M. Therefore, from now on, we take an atlas $\mathcal{O} = \{(O_\kappa, \varphi_\kappa)\}_{\kappa \in \Lambda}$ such that

$$\varphi_\kappa(U_\kappa) \subset \boldsymbol{R}^d_+ := \{(x^1, \ldots, x^d) \in \boldsymbol{R}^d : x^d > 0\}, \quad \varphi_\kappa(V_\kappa) \subset \boldsymbol{R}^{d-1} \times \{0\}$$

for $\kappa \in \Lambda_\partial$. Let $\{\omega_\kappa\}_{\kappa \in \Lambda}$ be a partition of unity subordinate to the atlas $\mathcal{O} = \{(O_\kappa, \varphi_\kappa)\}$:

$$\begin{aligned} &\omega_\kappa \in C^{2,\lambda}(\overline{D}), \quad \omega_\kappa \geq 0, \quad \operatorname{supp} \omega_\kappa \subset O_\kappa, \\ &\sum_{\kappa \in \Lambda} \omega_\kappa^2(x) = 1 \ \text{(for any } x \in \overline{D}). \end{aligned} \tag{3.1}$$

As before, we often use the same notation for denoting points of Euclidean space and the domain $\overline{D}$ when no confusion occurs. In particular, we often write x for $x_\kappa = \varphi_\kappa(x)$. We suitably extend the domain of the coefficients of $\mathcal{A}_\kappa$ to the whole space (i.e., $[0,\infty) \times \boldsymbol{R}^d$ for $\kappa \in \Lambda_\circ$ and $[0,\infty) \times \overline{\boldsymbol{R}^d_+}$ for $\kappa \in \Lambda_\partial$) without changing the values on $[0,\infty) \times \varphi_\kappa(\operatorname{supp} \omega_\kappa)$, and also extend the domain of the coefficients of $\mathcal{B}_\kappa$ to the whole space $[0,\infty) \times \partial \boldsymbol{R}^d_+$ in a similar way. Define the operator $\mathcal{L}_\kappa(s, x; \partial_s, \partial_x)$: if $\kappa \in \Lambda_\circ$

$$\mathcal{L}_\kappa(s, x; \partial_s, \partial_x) := \mathcal{A}_\kappa(s, x; \partial_s, \partial_x)$$

for $(s, x) \in [0,\infty) \times \boldsymbol{R}^d$, and if $\kappa \in \Lambda_\partial$

$$\mathcal{L}_\kappa(s, x; \partial_s, \partial_x) := 1_{R^d_+}(x)\mathcal{A}_\kappa(s, x; \partial_s, \partial_x) + 1_{\partial R^d_+}(x)\mathcal{B}_\kappa(s, x; \partial_x)$$

for $(s, x) \in [0,\infty) \times \overline{\boldsymbol{R}^d_+}$. We may assume that the coefficients of the operator $\mathcal{L}_\kappa$ for $\kappa \in \Lambda_\circ$ (resp. $\kappa \in \Lambda_\partial$) satisfy the condition of (L–1) over $[0,\infty) \times \boldsymbol{R}^d$ (resp. $[0,\infty) \times \overline{\boldsymbol{R}^d_+}$). According to $\kappa \in \Lambda_\circ$ or $\kappa \in \Lambda_\partial$, we take (τ, η) in $[0,\infty) \times \boldsymbol{R}^d$ or in $[0,\infty) \times \overline{\boldsymbol{R}^d_+}$. Define $\tilde{\mathcal{L}}_\kappa(\tau, \eta; \partial_s, \partial_x)$ from the operator $\mathcal{L}_\kappa(s, x; \partial_s, \partial_x)$ by freezing the coefficients of the main terms at (τ, η) and by deleting the lower order terms. For $\kappa \in \Lambda_\circ$ (resp. $\kappa \in \Lambda_\partial$), denote by $G^{\tau,\eta}_\kappa(s, x_\kappa; t, y_\kappa)$ (resp. $H^{\tau,\eta}_\kappa(s, x_\kappa;\, t, y_\kappa)$) the fundamental solution to the terminal value problem for $\tilde{\mathcal{L}}_\kappa(\tau, \eta;\, \partial_s, \partial_x)u = 0$. We refer to Ladyzenskaja et al [9] for the explicit form and necessary estimates of $G^{\tau,\eta}_\kappa(s, x; t, y)$. Let

$$P^{\tau,\eta}_\kappa(s, x_\kappa; t, y_\kappa) \;:=\; \begin{cases} G^{\tau,\eta}_\kappa(s, x_\kappa; t, y_\kappa) & \text{if} \quad \kappa \in \Lambda_\circ\,, \\ H^{\tau,\eta}_\kappa(s, x_\kappa; t, y_\kappa) & \text{if} \quad \kappa \in \Lambda_\partial\,, \end{cases}$$

$$p_\kappa^{t,y}(s,x;t,y) \quad := \quad \omega_\kappa(x) P_\kappa^{t,y_\kappa}(s,x_\kappa;t,y_\kappa)\omega_\kappa(y),$$

and

$$p^{t,y}(s,x;t,y) := \sum_{\kappa\in\Lambda} p_\kappa^{t,y}(s,x;t,y).$$

From now on, we denote $P_\kappa^{t,y}(s,x;\ t,y)$, $p_\kappa^{t,y}(s,x;\ t,y)$ and $p^{t,y}(s,x;t,y)$, by $\overline{P}_\kappa(s,x;t,y)$, $\overline{p}_\kappa(s,x;t,y)$ and $\overline{p}(s,x;t,y)$, respectively, and take $\overline{p}(s,x;t,y)$ as a *first parametrix*.

In the sequel, we need an orthogonal decomposition of the vector field $\upsilon(s,x)$ with respect to the Riemannian metric $a^\dagger(s)$ determined by the inverse $a(s,x)^{-1} = (a_{\lambda,jk}(s,x))$ of the tensor field $a(s,x)$. The orthogonal decomposition is used to formulate the second integral equation for getting a *pre-fundamental solution* (i.e. a function satisfying the conditions 1, 2, 3 of Definition 3.1). Denote by $\dfrac{\partial}{\partial n(s,x)}$ the Riemannian conormal derivative along ∂D with respect to $a^\dagger(s)$: for $(s,x)\in[0,\infty)\times V_\kappa$,

$$\frac{\partial}{\partial n(s,x)} = \frac{1}{\sqrt{a_\kappa^{dd}(s,x)}}\sum_{j=1}^{d} a_\kappa^{dj}(s,x)\frac{\partial}{\partial x_\kappa^j};$$

this is $(\lambda/2,\lambda)$–Hölder continuous in (s,x) of $[0,\infty)\times\partial D$. Then the orthogonal decomposition of $\upsilon(s,x)$ is as follows:

$$\upsilon(s,x) = \upsilon_n^\perp(s,x)\frac{\partial}{\partial n(s,x)} + \upsilon^\top(s,x)$$

with $\upsilon_n^\perp(s,x) > 0$ and $\upsilon^\top(s,x)\in T_x\partial D$. The scalar field $\upsilon_n^\perp(s,x)$ and the vector field $\upsilon^\top(s,x)$ are $(\lambda/2,\lambda)$–Hölder continuous in (s,x) of $[0,\infty)\times\partial D$.

Let us introduce two formal integrals for real-valued functions $f(\eta)$ on D and $g(\eta)$ on ∂D, respectively:

$$\int_D f(\eta)d^\circ\eta := \sum_{\kappa\in\Lambda}\int_{\varphi_\kappa(U_\kappa)} f(\eta_\kappa)d\eta_\kappa$$

and

$$\int_{\partial D} g(\eta)d_\tau^\partial\eta := \sum_{\kappa\in\Lambda_\partial}\int_{\varphi_\kappa(V_\kappa)} g(\eta_\kappa)\Omega_\kappa(\tau,\eta_\kappa)\sigma(d\eta_\kappa),$$

where we set $\sigma(d\eta_\kappa) = d\tilde{\eta}_\kappa\delta_0(d\eta_\kappa^d)$ for $\eta_\kappa = \left(\eta_\kappa^1,\ldots,\eta_\kappa^{d-1},\eta_\kappa^d\right)\equiv\left(\tilde{\eta}_\kappa,\eta_\kappa^{d-1}\right)$ and

$$\Omega_\kappa(\tau,\eta) := \frac{\omega_\kappa^2(\eta)}{2\upsilon_n^\perp(\tau,\eta)}\left\{\sum_{\ell\in\Lambda_\partial}\frac{\omega_\ell^2(\eta)}{\sqrt{a_\ell^{dd}(\tau,\eta)}}\det\left(\frac{\partial x_\kappa}{\partial x_\ell}(\eta)\right)\right\}^{-1}$$

for $\tau \geq 0, \eta \in \partial D$.

Let

$$\begin{aligned} \overline{q}_\kappa(s,x;t,y) &:= \mathcal{A}_\kappa(s,x;\partial_s,\partial_x)\overline{p}_\kappa(s,x;t,y), \\ \overline{q}(s,x;t,y) &:= \sum_{\kappa\in\Lambda} \overline{q}_\kappa(s,x;t,y), \end{aligned}$$

for $0 \leq s < t,\ x \in D$ and $y \in \overline{D}$; and consider the integral equation for an unknown function $\phi(s,x;t,y)$ $(0 \leq s < t,\ x \in D,\ y \in \overline{D})$:

$$\phi(s,x;t,y) = \overline{q}(s,x;t,y) + \int_s^t d\tau \int_D \overline{q}(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d^\circ\eta. \tag{3.2}$$

Using the solution $\phi(s,x;t,y)$, we set

$$\tilde{p}_\kappa(s,x;t,y) := \overline{p}_\kappa(s,x;t,y) + \int_s^t d\tau \int_D \overline{p}_\kappa(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d^\circ\eta.$$

Then

$$\mathcal{A}_\kappa(s,x;\partial_s,\partial_x)\tilde{p}_\kappa(s,x;t,y) = 0$$

for $0 \leq s < t,\ x \in D$ and $y \in \overline{D}$. Therefore we define a *second parametrix* $\tilde{p}(s,x;t,y)$ in the following way:

$$\tilde{p}(s,x;t,y) := \sum_{\kappa\in\Lambda} \tilde{p}_\kappa(s,x;t,y),$$

that is,

$$\tilde{p}(s,x;t,y) = \overline{p}(s,x;t,y) + \int_s^t d\tau \int_D \overline{p}(s,x;\tau,\eta)\phi(\tau,\eta;t,y)d^\circ\eta.$$

It satisfies

$$\mathcal{A}(s,x;\partial_s,\partial_x)\tilde{p}(s,x;t,y) = 0$$

for $0 \leq s < t,\ x \in D$ and $y \in \overline{D}$.

Next define

$$\begin{aligned} \tilde{q}_\kappa(s,x;t,y) &:= \mathcal{B}_\kappa(s,x;\partial_x)\tilde{p}_\kappa(s,x;t,y), \\ \tilde{q}(s,x;t,y) &:= \sum_{\kappa\in\Lambda_\partial} \tilde{q}_\kappa(s,x;t,y) \end{aligned}$$

for $0 \le s < t,\ x \in \partial D$ and $y \in \overline{D}$. Let us consider the integral equation for an unknown function $\psi(s,x;t,y)$ $(0 \le s < t,\ x \in \partial D,\ y \in \overline{D})$:

$$\psi(s,x;t,y) = \tilde{q}(s,x;t,y) + \int_s^t d\tau \int_{\partial D} \tilde{q}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)d_\tau^\partial \eta. \tag{3.3}$$

If we set

$$\hat{p}(s,x;t,y) := \tilde{p}(s,x;t,y) + \int_s^t d\tau \int_{\partial D} \tilde{p}(s,x;\tau,\eta)\psi(\tau,\eta;t,y)d_\tau^\partial \eta,$$

then

$$\mathcal{B}(s,x;\partial_x)\hat{p}(s,x;t,y) = 0$$

for $0 \le s < t,\ x \in \partial D$ and $y \in \overline{D}$; and further

$$\mathcal{L}(s,x;\partial_s,\partial_x)\hat{p}(s,x;t,y) = 0$$

for $0 \le s < t$ and $x, y \in \overline{D}$. That is, $\hat{p}(s,x;t,y)$ becomes a pre-fundamental solution. Then we modify $\hat{p}(s,x;t,y)$ so as to be satisfied the fourth condition of Definition 3.1: Let

$$\rho(y) := \sum_{\kappa \in \Lambda} \omega_\kappa^2(y)\sqrt{\det(g_{\kappa,jk}(y))}, \tag{3.4}$$

where $g_{\kappa,jk} := g\left(\partial/\partial x_\kappa^j, \partial/\partial x_\kappa^k\right)$ is the element of the Riemannian metric tensor field. Then a fundamental solution $p(s,x;t,y)$ is given by

$$p(s,x;t,y) = \hat{p}(s,x;t,y)\rho^{-1}(y).$$

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