International Journal of Pure and Applied Mathematics
Volume 21 No. 2 2005, 267-274

SECOND ORDER GENERALIZED LINEAR SYSTEMS. A GEOMETRIC APPROACH

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Abstract: Let (E, A_1, A_2, B) be a quadruple of matrices representing a two-order generalized time-invariant linear system, $E\ddot{x} = A_1\dot{x} + A_2x + Bu$. We study the controllability character under an algebraic point of view.

AMS Subject Classification: 15A21, 93B52
Key Words: two-order generalized linear systems, feedback, controllability

1. Introduction

A second order generalized linear system is described by the following state space equation

$$E\ddot{x} = A_1\dot{x} + A_2x + Bu, \tag{1}$$

where A_i are n-square complex matrices and B a $n \times m$-rectangular complex matrix in adequate size. We denote this kind of systems by quadruples of matrices (E, A_1, A_2, B), and the space of all quadruples by $\mathcal{M}$.

In this paper, and using order-reduction process generalizing the one given in Lancaster and Tismenestsky [5], for ℓ-order linear systems, we present necessary conditions for existence of a control w in such a way the state can be

Received: May 5, 2005
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driven from any position to any other in a prescribed period of time. Some sufficient conditions for $E = I$ are presented in Clotet and García-Planas [2].

The structure of this paper is as follows.

In Section 2, an equivalence relation over the space of second order generalized linear systems is defined and it induce an equivalence over the space of order-reduced generalized systems. We observe that the equivalent order-reduced generalized systems are feedback and derivative feedback equivalent as linear systems, but the converse is not true.

In Section 3, the controllability analysis relating controllability of second order generalized linear systems and controllability of linear systems associated is presented.

2. Orbits of Order-Reduced Generalized Systems

Let $E\ddot{x} = A_1\dot{x} + A_2x + Bu$ be a second order generalized linear system as in the introduction, the standard transformations that can be applied are basis change in the state space, basis change in the input space, feedback, derivative feedback, second order derivative feedback. Then, the initial equation is transformed to

$$(P^{-1}EP + P^{-1}BF_3)\ddot{x} = (P^{-1}A_1P + P^{-1}BF_1)\dot{x} + (P^{-1}A_2P + P^{-1}BF_2)x + P^{-1}BQu.$$

This leads to the definition of the following equivalence relation in the space $\mathcal{M}$.

Definition 1. Two quadruples (E, A_1, A_2, B), $(E', A'_1, A'_2, B') \in \mathcal{M}$, are equivalent if and only if there exist matrices $P \in Gl(n;\mathbb{C})$, $Q \in Gl(m;\mathbb{C})$ and $F_1, F_2, F_3 \in M_{m\times n}(\mathbb{C})$ such that these equalities $E' = P^{-1}EP + P^{-1}BF_3$, $A'_1 = P^{-1}A_1P + P^{-1}BF_1, A'_2 = P^{-1}A_2P + P^{-1}BF_2, B' = P^{-1}BQ$ hold.

It is straightforward that this relation is an equivalence relation.

To find a canonical reduced form for quadruples of matrices under this equivalence relation is an open problem. In order to obtain some structural invariants we consider the order-reduction process.

We consider $X = \begin{pmatrix} x & \dot{x} \end{pmatrix}^t$, then, we can rewrite the second order generalized linear system (1) as

$$\mathbb{E}\dot{X} = \mathbb{A}X + \mathbb{B}u, \tag{2}$$

with $\mathbb{E} = \begin{pmatrix} I_n & 0 \\ 0 & E \end{pmatrix}$, $\mathbb{A} = \begin{pmatrix} 0 & I_n \\ A_2 & A_1 \end{pmatrix}$ and $\mathbb{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}$.

This expression permits to consider feedback and derivative feedback equivalence relation. Remember that (see Carriegos and García-Planas [1], for example), two generalized linear systems $(\mathbb{E}, \mathbb{A}, \mathbb{B})$ and $(\mathbb{E}_1, \mathbb{A}_1, \mathbb{B}_1)$ are called feedback and derivative feedback equivalent if and only if there exist $(\mathbb{P}, \mathbb{Q}, \mathbb{F}_{\mathbb{E}}, \mathbb{F}_{\mathbb{A}})$

in the full group $\mathcal{G} = \{(\mathbb{P}, \mathbb{Q}, \mathbb{F}_{\mathbb{E}}, \mathbb{F}_{\mathbb{A}}) \mid \mathbb{P} \in \mathrm{Gl}(2n; \mathbb{C}), \mathbb{Q} \in \mathrm{Gl}(m; \mathbb{C}), \mathbb{F}_{\mathbb{E}}, \mathbb{F}_{\mathbb{A}} \in M_{m\times 2n}(\mathbb{C})$ such that

$$\begin{pmatrix} \mathbb{E}_1 & \mathbb{A}_1 & \mathbb{B}_1 \end{pmatrix} = \mathbb{P}^{-1} \begin{pmatrix} \mathbb{E} & \mathbb{A} & \mathbb{B} \end{pmatrix} \begin{pmatrix} \mathbb{P} & 0 & 0 \\ 0 & \mathbb{P} & 0 \\ \mathbb{F}_{\mathbb{E}} & \mathbb{F}_{\mathbb{A}} & \mathbb{Q} \end{pmatrix}.$$

It is easy to prove the following proposition.

Proposition 1. *Let (E, A_1, A_2, B) and (E', A'_1, A'_2, B') be two equivalent quadruples of matrices in $\mathcal{M}$. Then, the order-reduced generalized systems $(\mathbb{E}, \mathbb{A}, \mathbb{B})$, $(\mathbb{E}', \mathbb{A}', \mathbb{B}')$, are feedback and derivative feedback equivalent.*

Notice that the converse is not true.

Before proposition ensures that all structural invariants of a order-reduced system as a triple under feedback and derivative feedback equivalence are invariants for a given quadruple, but the set of these invariants is not a complete system of invariants.

In order to preserve the form (2) for equivalent generalized linear systems, in the sense that the only equivalent triples are those that are order reduced generalized of some equivalent second order generalized linear system, we need to restrict to the subgroup $\mathcal{G}_2 \subset \mathcal{G}$ formed by matrices $(\mathbb{P}, \mathbb{Q}, \mathbb{F}) \in \mathcal{G}$ with $\mathbb{P} = \mathrm{diagonal}(P, P)$, $P \in \mathrm{Gl}(n, \mathbb{C})$, and $\mathbb{F}_{\mathbb{E}} = \begin{pmatrix} 0 & F_E \end{pmatrix}$, with $F_E \in M_{m\times n}(\mathbb{C})$. Then we have the following proposition.

Proposition 2. *Two quadruples (E, A_1, A_2, B) and $(E', A'{}_1, A'{}_2, B')$ in $\mathcal{M}$ are equivalent, if and only if the triples $(\mathbb{E}, \mathbb{A}, \mathbb{B})$ and $(\mathbb{E}', \mathbb{A}', \mathbb{B}')$ are $\mathcal{G}_2$-equivalent.*

Written in a matrix form

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & I_n & 0 \\ E & A_2 & A_1 & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ F_3 & F_2 & F_1 & Q \end{pmatrix} = \begin{pmatrix} 0 & 0 & I_n & 0 \\ E' & A'_2 & A'_1 & B' \end{pmatrix}.$$

So, we have the following definition.

Definition 2. Two second order generalized linear systems (E', A'_1, A'_2, B'), $(E'', A''_1, A''_2, B'') \in \mathcal{M}$, are equivalent if and only if the associated order reduced generalized systems $(\mathbb{E}', \mathbb{A}', \mathbb{B}')$, $(\mathbb{E}'', \mathbb{A}'', \mathbb{B}'')$ are $\mathcal{G}_2$-equivalent.

3. Controllability

We recall that a second order generalized linear system is called controllable if, for any $t_1 > 0$, $x(0), \dot{x}(0) \in \mathbb{C}^n$ and $w, w_1 \in \mathbb{C}^n$, there exists a control $u(t)$ such that $x(t_1) = w$, $\dot{x}(t_1) = w_1$. This definition is a natural generalization of controllability concept in the first order linear systems.

Taking into account that $x(t)$ is a solution of the second order generalized linear system if and only if $\begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^t$ is a solution of the associated order-reduced generalized system, we have that the second order generalized linear system is controllable if and only if the order-reduced generalized system is controllable.

So, we can use results about controllability of generalized linear systems, in particular (see Dai [3], Carriegos and Garcí-Planas [1], for example) we have that the triple $(\mathbb{E}, \mathbb{A}, \mathbb{B})$ (order-reduced generalized of (E, A_1, A_2, B)) is controllable if and only if

$$i)\quad \operatorname{rank}\begin{pmatrix} \mathbb{E} & \mathbb{B} \end{pmatrix} = 2n\,; \qquad ii)\quad \operatorname{rank}\begin{pmatrix} s\mathbb{E} - \mathbb{A} & \mathbb{B} \end{pmatrix} = 2n\,, \quad \forall s \in \mathbb{C}\,. \tag{3}$$

Remark 1. Condition i) ensures that there exists a derivative feedback $\mathbb{F}_{\mathbb{E}} \in M_{m\times 2n}(\mathbb{C})$ such that $\mathbb{E} + \mathbb{B}\mathbb{F}_{\mathbb{E}}$ is regular and the system is standardizable, so there exists a second order derivative feedback F_3 (the matrix formed by the n last columns of $\mathbb{F}_E$) such that $E + BF_3$ is regular and the second order generalized linear system is standardizable.

Making elementary transformations in the matrices on (3), we can analyze the controllability directly from the matrices defining the second order generalized linear system, obtaining the following characterization.

Theorem 1. *The second order generalized linear system* (E, A_1, A_2, B), *is controllable if and only if*

$$i)\quad \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n\,; \quad ii)\quad \operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = n\,, \quad \forall s \in \mathbb{C}\,.$$

Proof. $i)$ $\operatorname{rank}\begin{pmatrix} \mathbb{E} & \mathbb{B} \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix}$.

$$ii)\qquad \operatorname{rank}\begin{pmatrix} s\mathbb{E} - \mathbb{A} & \mathbb{B} \end{pmatrix} = \operatorname{rank}\left(s\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} 0 & I \\ A_2 & A_1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ B \end{pmatrix}\right)$$
$$= \operatorname{rank}\begin{pmatrix} 0 & I & 0 \\ s^2E - sA_1 - A_2 & 0 & B \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix}.$$

Theorem 1, permits us to define controllability in the following manner.

Definition 3. We say that the second order generalized linear system (E, A_1, A_2, B), is controllable if and only if

$$i)\quad \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n\,; \qquad ii)\quad \operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = n\,, \quad \forall s \in \mathbb{C}.$$

It is well known that, the controllability of a linear system is invariant under feedback equivalence, then the controllability of order reduced generalized systems is invariant under $\mathcal{G}_2$-equivalence. So, the controllability of second order generalized linear systems is invariant under considered equivalence relation. In fact we have the following proposition.

Proposition 3. *The controllability condition is invariant under equivalence defined before.*

Proof. Let (E, A_1, A_2, B) and (E', A'_1, A'_2, B') be two equivalent quadruples. Then

$$\begin{pmatrix} E' & B' \end{pmatrix} = P^{-1} \begin{pmatrix} E & B \end{pmatrix} \begin{pmatrix} P & 0 \\ F_3 & Q \end{pmatrix},$$

$$\begin{pmatrix} s^2E' - sA'_1 - A'_2 & B' \end{pmatrix} = P^{-1} \begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} \begin{pmatrix} P & 0 \\ s^2F_3 - sF_1 - F_2 & Q \end{pmatrix}. \quad \square$$

4. Geometric Approach

In this section, we are dealing with quadruples of matrices as triples of linear maps defined modulo a subspace $(f, g, h) : X \longrightarrow X/W$, where X is a finite dimensional vector space, W is a linear subspace verifying $f_{|W} = g_{|W} = h_{|W}$, generalizing results for a linear map defined modulo a subspace, or pair of linear maps defined modulo a subspace (see García-Planas [4] for example).

Definition 4. *Let $(f, g, h) : X \longrightarrow X/W$ be a triple of linear maps defined modulo a subspace. We consider the following triples of linear maps induced in a natural way by (f, g, h):*

$$\begin{array}{cc} (\dot{f}, \dot{g}, \dot{h}) : W \to W_1 & (f_1, g_1, h_1) : X/W \to X_1/W_1 \\ w \to f(w) = g(w) = h(w) & \pi(x) \to \pi_1(f, g, h)(x) \end{array} \tag{4}$$

where $W_1 = f(W) = g(W) = h(W)$, $X_1 = X/W$ and $\pi : X \longrightarrow X/W$ and $\pi_1 : X/W \longrightarrow X_1/W_1$ the canonical projections.

We will call simply triple of linear maps.

Then we have the following commutative diagrams:

$$\begin{array}{ccc} W & \xrightarrow{\dot{f}} & W_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X/W = X_1 \\ \downarrow & & \downarrow \\ X/W & \xrightarrow{f_1} & X_1/W_1 = X_2 \end{array} \quad \begin{array}{ccc} W & \xrightarrow{\dot{g}} & W_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X/W = X_1 \\ \downarrow & & \downarrow \\ X/W & \xrightarrow{g_1} & X_1/W_1 = X_2 \end{array} \quad \begin{array}{ccc} W & \xrightarrow{\dot{h}} & W_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & X/W = X_1 \\ \downarrow & & \downarrow \\ X/W & \xrightarrow{h_1} & X_1/W_1 = X_2 \end{array} \tag{5}$$

Notice that the maps $\dot{f} = \dot{g} = \dot{h}$ are surjective, $\dim X_1 \leq \dim X$ and $\dim X_1 = \dim X$ if and only if $W = \{0\}$.

Let $(f, g, h) : X \longrightarrow X/W$ a triple of linear maps. In order to obtain a matrix representation we consider triples of bases of X adapted to W, that is bases ($b_f = \{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+m}\}$, $b_g = \{\bar{e}_1, \dots, \bar{e}_n, e_{n+1}, \dots, e_{n+m}\}$, $b_h = \{\tilde{e}_1, \dots, \tilde{e}_n, e_{n+1}, \dots, e_{n+m}\}$,) such that $\{e_{n+1}, \dots, e_{n+m}\}$ is a base for W, and $e_i - \bar{e}_i, e_i - \tilde{e}_i \in W$ for all $i = 1, \cdots, n$. Consequently $\{e_1 + W, \dots, e_n + W\} = \{\bar{e}_1 + W, \dots, \bar{e}_n + W\} = \{\tilde{e}_1 + W, \dots, \tilde{e}_n + W\}$ is a base for X/W.

Associated matrices $\mathbf{A}_f$, $\mathbf{A}_g$ and $\mathbf{A}_h$ of the linear maps f, g and h in this triple of adapted bases are in the form

$$\mathbf{A}_f = \begin{pmatrix} E & B \end{pmatrix}, \quad \mathbf{A}_g = \begin{pmatrix} A_1 & B \end{pmatrix}, \quad \mathbf{A}_h = \begin{pmatrix} A_2 & B \end{pmatrix}, \tag{6}$$

with $E, A_1, A_2 \in M_n(\mathbb{C})$, $B \in M_{n \times m}(\mathbb{C})$.

We will write simply as a quadruple of matrices (E, A_1, A_2, B).

In order to see equivalence of quadruples as an equivalence of triples of linear maps $(f, g, h) : X \longrightarrow X/W$, $(f', g', h') : X' \longrightarrow X'/W'$ we consider the triples of isomorphisms $(\varphi, \psi, \phi) : X \longrightarrow X'$, where the maps induced in a natural way

$$(\dot{\varphi}, \dot{\psi}, \dot{\phi}) : W \longrightarrow W' \quad (\tilde{\varphi}, \tilde{\psi}, \tilde{\phi}) : X/W \longrightarrow X'/W' \tag{7}$$

verify $\dot{\varphi} = \dot{\psi} = \dot{\phi}$ and $\tilde{\varphi} = \tilde{\psi} = \tilde{\phi}$. We denote by $\mathcal{H}(W)$ the group of such pairs of isomorphisms, obviously we must suppose $\dim X = \dim X'$ and $\dim W = \dim W'$. From now on, these dimensions will be denoted by $n + m$ and m respectively.

Definition 5. Let $(f, g, h) : X \longrightarrow X/W$, $(f', g', h') : X' \longrightarrow X'/W'$ be two triples of linear maps. We say that they are equivalent (written $(f, g, h) \sim (f', g', h')$), if there is $(\varphi, \psi, \phi) \in \mathcal{H}(W)$ such that $f' \circ \varphi = \tilde{\varphi} \circ f, g' \circ \psi = \tilde{\psi} \circ g, h' \circ \phi = \tilde{\phi} \circ h$ and we will write simply as

$$(f', g', h') \circ (\varphi, \psi, \phi) = (\tilde{\varphi}, \tilde{\psi}, \tilde{\phi}) \circ (f, g, h)\,. \tag{8}$$

Proposition 4. *Let (E, A_1, A_2, B) and (E', A'_1, A'_2, B') be two quadruples of matrices corresponding to the matrix representation of two equivalent triples of maps (f, g, h) and (f', g', h') respectively. Then there exist invertible matrices $P \in \mathrm{Gl}(n; \mathbb{C})$, $Q \in \mathrm{Gl}(m; \mathbb{C})$ and rectangular matrices $F_1, F_2, F_3 \in M_{m \times n}(\mathbb{C})$*

such that the following equality holds.

$$\begin{pmatrix} E' & A_1' & A_2' & B' \end{pmatrix} = P^{-1} \begin{pmatrix} E & A_1 & A_2 & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ F_3 & F_1 & F_2 & Q \end{pmatrix}. \tag{9}$$

Proof. It suffices to observe the form of matrices of automorphisms of $\mathbb{C}^{n+m} \simeq \mathbb{C}^n \times \mathbb{C}^m$ applying $\{0\} \times \mathbb{C}^m$ in $\{0\} \times \mathbb{C}^m$ and $\mathbb{C}^{n+m}/\mathbb{C}^m$ in $\mathbb{C}^{n+m}/\mathbb{C}^m$. □

Proposition 5. *Let (f, g, h) be a triple of linear maps with* $\dim W = 1$, *f a linear map defined modulo a subspace corresponding to a controllable map. Then there exists a triple of adapted bases (b_f, b_g, b_h) such that*

$$\mathbf{E} = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \ \mathbf{A}_1 = (a^1_{i,j}), \ \mathbf{A}_2 = (a^2_{i,j}), \ \mathbf{B} = \begin{pmatrix} 0_{n-1} \\ 1 \end{pmatrix},$$

with $a^1_{n,j} = a^2_{n,j} = 0$

Proposition 6. *Numbers a^1_{ij} in matrix $\mathbf{A}_1$ and a^2_{ij} in matrix $\mathbf{A}_2$ characterize the equivalence class of quadruples of matrices.*

Now, it is easy to obtain conditions for controllability of a quadruple of matrices representing a second order generalized system.

Theorem 2. *Let E, A_1, $A_2 \in M_n(\mathbb{C})$ and $B \in M_{n\times 1}(\mathbb{C})$, $n > 1$. Let (E, B) a controllable pair of matrices. Let (b_f, b_g, b_h) an adapted triple of bases such that the quadruple takes the form given in Proposition 5.*

A necessary condition for controllability of the quadruple is

$$\begin{pmatrix} A_1 & B \end{pmatrix} e_1 \notin [e_n + W] \quad \text{or} \quad \begin{pmatrix} A_2 & B \end{pmatrix} e_1 \notin [e_n + W]$$

Proof. It suffices to compute the bases b_f. □

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