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# ON THE FÖLLMER-SCHWEIZER-BARLOW MODEL

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**Abstract:** We consider the Föllmer-Schweizer-Barlow microeconomic model, characterize the no-arbitrage situation, analyse the spectrum of the underlying process, and provide a series expansion for the associated option prices.

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## 1. Model Description and No-Arbitrage

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$ be a minimal risk-neutral filtered probability space and consider a linear Ornstein-Uhlenbeck (OU) process $\{Y_t\}_{t\in\mathbb{R}_+}$ with positive parameters $(k, a, \sigma)$, given by $dY_t = -k(Y_t - a)dt + \sigma dW_t$, where $Y_0$ is a given real number, $\{W_t\}_{t\in\mathbb{R}_+}$ is a standard Brownian motion, and $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is the Brownian filtration. The microeconomic model considered in [5] and [2] is given by

$$X_t := f_\alpha(Y_t), \text{ where } f_\alpha(x) = \begin{cases} (1+\alpha x)^{1/\alpha} \text{ for } \alpha \neq 0\,, \\ e^x \text{ for } \alpha = 0, \end{cases} \tag{1.1}$$

and with $Y_t$ truncated at some maximum level (if $\alpha < 0$) or minimum level (if $\alpha > 0$) for the above composition to make sense. Indeed, $f_\alpha$ is undefined for values of $x$ with $1 + \alpha x < 0$ (except for a few special values $\alpha$). So, the

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process $\{X_t\}_{t\in\mathbb{R}_+}$ is only well defined if $Y_t \geq -1/\alpha$, $\alpha > 0$ or $Y_t < -1/\alpha$, $\alpha < 0$. For instance, if $\alpha = 1$ then $\{X_t\}_{t\in\mathbb{R}_+}$ is an OU with parameters $(k, 1+a, \sigma)$, and if $\alpha = 0$ then $\{X_t\}_{t\in\mathbb{R}_+}$ is the exponential of an OU with parameters $\big(k, a+\sigma^2/2k, \sigma\big)$.

**Lemma 1.** *$\{X_t\}_{t\in\mathbb{R}_+}$ is a time homogeneous diffusion process with state space $(0,+\infty)$, and whose dynamics are driven by the following SDE*

$$dX_t = b(X_t)dt + \sigma X_t^{1-\alpha} dW_t, \tag{1.2}$$

*where the drift parameter is given by*

$$b(x) = \begin{cases} -\dfrac{k}{\alpha}x + \left(\dfrac{k}{\alpha} + ka\right)x^{1-\alpha} + \dfrac{1-\alpha}{2}\sigma^2 x^{1-2\alpha} \text{ if } \alpha \neq 0, \\ -kx\ln x + \left(ka + \dfrac{\sigma^2}{2}\right)x \text{ if } \alpha = 0. \end{cases} \tag{1.3}$$

*The boundaries are as follows: if $\alpha > 0$ then $0$ is regular and $+\infty$ is natural, if $\alpha < 0$ then $0$ is natural and $+\infty$ is regular, and if $\alpha = 0$ then both $0$ and $+\infty$ are natural. Any other number in $(0,+\infty)$ is regular for any value of $\alpha$. In particular $\{X_t\}_{t\in\mathbb{R}_+}$ is time reversible and has a stationary distribution on $(0,+\infty)$ for any value of $\alpha\in\mathbb{R}$.*

*Proof.* First of all, remark that $f(-\infty) = 0$ and $f(-1/\alpha) = +\infty$ if $\alpha < 0$, $f(-1/\alpha) = 0$ and $f(+\infty) = +\infty$ if $\alpha > 0$, $f(-\infty) = 0$ and $f(+\infty) = +\infty$ if $\alpha = 0$, hence the state space of $X_t$ is $(0,+\infty)$ for any value of $\alpha$. As $f_\alpha$ is strictly increasing and $f''_\alpha$ is continuous on their common domain, by [10], Theorem 2.1, $\{X_t\}_{t\in\mathbb{R}_+}$ satisfies the following SDE: $dX_t = b(X_t)dt + \sigma f'_\alpha\Big(f_\alpha^{-1}(X_t)\Big)dW_t$, where $f_\alpha^{-1}$ is the inverse of $f_\alpha$, and the drift parameter is given by

$$b(x) = -kf'_\alpha\Big(f_\alpha^{-1}(x)\Big)f_\alpha^{-1}(x) + kaf'_\alpha\Big(f_\alpha^{-1}(x)\Big) + \sigma^2 f''_\alpha\Big(f_\alpha^{-1}(x)\Big)/2\,.$$

Straightforward computations on $f_\alpha$ give the SDE (1.2) in the conclusion.

Let $x_0$ be any (fixed) number in $(0,+\infty)$, and introduce the scale function $S(x)$, its derivative $s(x)$, the speed function $M(x)$ and its derivative $m(x)$ by

$$\begin{aligned} S(x) &= \int_{x_0}^{x} s(y)dy = \int_{x_0}^{x} \exp\left(-\int_{x_0}^{y}\frac{2b(t)}{\sigma^2 t^{2-2\alpha}}dt\right)dy, \\ M(x) &= \int_{x_0}^{x} m(y)dy = \int_{x_0}^{x}\frac{1}{\sigma^2 y^{2-2\alpha}s(y)}dy \end{aligned}$$

for $0 < x < +\infty$. The other two functionals we need to consider are:

$$\Sigma(x) = \int_{x_0}^{x} [S(t) - S(x)]dM(t) = \int_{x_0}^{x} [M(x_0) - M(t)]dS(t),$$
$$N(x) = \int_{x_0}^{x} [S(x_0) - S(t)]dM(t) = \int_{x_0}^{x} [M(t) - M(x)]dS(t).$$

The conclusion on the nature of boundaries follows (see [10], p. 226) if we prove the following.

For $\alpha < 0$:

$$\begin{cases} S(0) = -\infty,\ M(0) < \infty,\ \Sigma(0) = \infty,\ N(0) = \infty, \\ 0 < S(+\infty) < \infty,\ M(+\infty) < \infty,\ \Sigma(+\infty) < \infty,\ N(+\infty) < \infty. \end{cases}$$

For $\alpha > 0$:

$$\begin{cases} -\infty < S(0) < 0,\ M(0) < \infty,\ \Sigma(0) < \infty,\ N(0) < \infty, \\ S(+\infty) = \infty,\ M(+\infty) < \infty,\ \Sigma(+\infty) = \infty,\ N(+\infty) = \infty. \end{cases}$$

For $\alpha = 0$:

$$\begin{cases} S(0) = -\infty,\ M(0) < \infty,\ \Sigma(0) = \infty,\ N(0) = \infty, \\ S(+\infty) = \infty,\ M(+\infty) < \infty,\ \Sigma(+\infty) = \infty,\ N(+\infty) = \infty. \end{cases}$$

All the above is based on the following closed-form formula:

$$s(x) = \begin{cases} x^{\alpha-1} \exp\left[\dfrac{k(x^{2\alpha}-1) - 2k(1+a\alpha)(x^{\alpha}-1)}{\alpha^2\sigma^2}\right] \text{ if } \alpha \neq 0 \\ x^{-(1+2ka/\sigma^2)} \exp\left(\dfrac{k}{\sigma^2}\ln^2 x\right) \text{ if } \alpha = 0, \end{cases} \tag{1.4}$$

and standard manipulations of type I and II improper integrals. The fact that $\{X_t\}_{t\in\mathbb{R}_+}$ has a stationary distribution on $(0,+\infty)$ follows (see [10], p. 220-221) from the integrability of $m(x)$ on $(0,\infty)$ for any value of $\alpha$. □

As a measure of market fairness, it is essential to decide whenever the underlying model has the no arbitrage property, that is, by betting on the process $\{X_t\}_{t\in\mathbb{R}_+}$ it is not possible to make a profit with zero net investment and without bearing any risk. In probabilistic terminology, the lack of riskless gain amounts to studying the existence of an equivalent local martingale measure for $\{X_t\}_{t\in\mathbb{R}_+}$ on a finite time horizon.

**Proposition 2.** *The model (1.1) has the no-arbitrage property if and only if* $\alpha \leq 0$.

*Proof.* Denote the time horizon by $[0,T]$; the process $\{X_t\}_{t\in[0,T]}$ will be stopped when it reaches zero. If such an equivalent local martingale measure $Q$ exists, by the Cameron-Martin-Girsanov Theorem (see [9]) we have $dX_t = \sigma X_t^{1-\alpha} dW_t^Q$ under $Q$, for some $Q$-Brownian motion $\{W_t^Q\}_{t\in[0,T]}$, and where the process is stopped, as before, when it reaches zero. Hence the probabilities $P$ and $Q$ are equivalent on $\mathcal{F}_T$ (or they have the same support) if and only if, at time $T$, the Radon-Nikodym density processes satisfy $dP/dQ > 0$ ($Q-$a.s.) and $dQ/dP > 0$ ($P-$a.s.), that is, if and only if $\{X_t\}_{t\in[0,T]}$ cannot reach zero in finite time under both $P$ and $Q$ (as explained in [4], Theorem 2.1).

By Feller's criterion ([4], Section 2), $\{X_t\}_{t\in[0,T]}$ does not reach zero under $Q$ if

$$\int_0^1 \frac{x}{\sigma^2 x^{2(1-\alpha)}} dx = +\infty,$$

and the latter holds if and only if $\alpha \leq 0$. Since $S(0) = -\infty$ for $\alpha \leq 0$ (cf. the proof of Lemma 1), the process $\{X_t\}_{t\in[0,T]}$ stopped at an arbitrary level is a local martingale bounded above, and also a submartingale (bounded below a.s.), that is, $X_t$ cannot reach zero in finite time under $P$ either. Conversely, if $\alpha > 0$, then

$$\int_0^1 \frac{b^2(x)}{\sigma^2 x^{2(1-\alpha)}} dx = +\infty\,,$$

hence, using again Feller's criterion, we deduce that $P$ and $Q$ cannot be equivalent. □

**Remark.** The proof of Proposition 2 together with Theorem 1.1 in [3] show that, for $\alpha \leq 0$, the model (1.1) is incomplete, that is, there is an infinite number of equivalent local martingale measures, and also satisfies the stronger "no free lunch with vanishing risk" property.

## 2. Spectrum

We are interested in the local evolution of the diffusion process $\{X_t\}_{t\in\mathbb{R}_+}$, therefore we consider its infinitesimal generator $L$, defined for any value of $\alpha$ by

$$L\phi(x) = \frac{1}{2}\sigma^2 x^{2-2\alpha}\phi''(x) + b(x)\phi'(x).$$

The domain of $L$ consists of all functions $\phi$ such that $\phi$ and $L\phi$ are square integrable with respect to the stationary distribution of $X_t$, $\phi'$ is absolutely continuous and satisfies appropriate boundary conditions:

$$\lim_{x\searrow 0} \frac{\phi'(x)}{s(x)} = \lim_{x\nearrow +\infty} \frac{\phi'(x)}{s(x)} = 0,$$

(cf. [7], pp. 9-10), and where $s(x)$ is the derivative of the scale function, see formula (1.4).

In order to obtain a characterization of the spectrum of $L$, we consider the eigenvalue problem

$$\text{Find } \phi \text{ in the domain of } L \text{ such that } L\phi = -\lambda\phi \text{ for some } \lambda \in \mathbb{R}. \tag{2.1}$$

Although the drift and the diffusion parameters exhibit a singularity at $x = 0$ (for $\alpha > 1/2$ and, respectively, for $\alpha > 1$), the operator $L$ is negative semi-definite for any value of $\alpha$, as consequence of Lemma 1. In particular, from now on, in problem (2.1) we consider eigenvalues $\lambda \geq 0$ only.

For diffusion processes with natural boundaries (see [8], Theorem 4.2), there exists $0 \leq \lambda_0 \leq +\infty$ such that any eigenfunction $\phi$ of problem (2.1) crosses the zero axis a finite number of times if $\lambda < \lambda_0$, and an infinite number of times if $\lambda > \lambda_0$. In particular, when $\lambda_0 = +\infty$, the spectrum of $L$ is discrete; when $\lambda_0 = 0$, the only eigenfunction of $L$ is the constant function, and when $\lambda_0 > 0$, the spectrum exhibits a mixed behavior, changing from discrete ($\lambda < \lambda_0$) to continuous ($\lambda > \lambda_0$). By Lemma 1, the boundaries of $\{X_t\}_{t\in\mathbb{R}_+}$ are not always natural (unless $\alpha = 0$), hence we cannot apply directly Theorem 4.2 in [8]. However, inspired by a suggestion in Section 4.3 of the same paper, we obtain the following result that completely describes the nature of the spectrum of $L$.

**Theorem 3.** *If $\alpha \geq -1/a$ then $\{X_t\}_{t\in\mathbb{R}_+}$ has continuous spectrum. If $\alpha < -1/a$ then $\{X_t\}_{t\in\mathbb{R}_+}$ has discrete spectrum up to the critical value*

$$\lambda_0 = \frac{k^2(1 + a\alpha)^2}{2\alpha^2\sigma^2} \tag{2.2}$$

*of $\lambda$ in the eigenvalue problem (2.1), and beyond this critical value the spectrum becomes continuous.*

*Proof.* Let us consider the diffusion parameter of $\{X_t\}_{t\in\mathbb{R}_+}$ in its "natural scale" denoted by $\theta$, and given in implicit form by $\theta(S(x)) = \sigma x^{1-\alpha}s(x)$. By equation (1.4) we can easily see that, for any $x$,

$$\theta'(S(x)) = \begin{cases} \dfrac{2k[x^\alpha - (1 + a\alpha)]}{\alpha\sigma} & \text{if } \alpha \neq 0, \\ \dfrac{2k(\ln x - a)}{\sigma} & \text{if } \alpha = 0. \end{cases} \tag{2.3}$$

For $\alpha \neq 0$, fix $\bar{x}$ in $(0, +\infty)$ and consider two new processes with the same diffusion and drift coefficients, but one concentrated on $(0, \bar{x})$ and the other on

$(\bar{x},+\infty)$. Solutions $\phi$ to the eigenvalue problem (2.1) for the original process can be obtained by smoothly pasting together two solutions to the eigenvalue problem for the new processes. If both latter solutions only cross the zero axes a finite number of times, the same must be true for the pasted together solution; if at least one of them crosses the zero axis an infinite number of times, the same is true for the pasted solution. Define

$$\lambda_0 := \begin{cases} \lambda_0(\bar{x}) = \dfrac{1}{8}\min\Big\{[\theta'(S(0))]^2, [\theta'(S(\bar{x}))]^2, [\theta'(S(+\infty))]^2\Big\} \text{ if } \alpha \neq 0, \\ \dfrac{1}{8}\min\Big\{[\theta'(S(-\infty))]^2, [\theta'(S(+\infty))]^2\Big\} \text{ if } \alpha = 0. \end{cases} \tag{2.4}$$

When $\alpha \neq 0$ we apply the Sturm-Liouville theory to the diffusion processes concentrated on $(0,\bar{x})$ and $(\bar{x},+\infty)$, and when $\alpha = 0$ we apply it to $\{X_t\}_{t\in\mathbb{R}_+}$ itself, to obtain that a solution $\phi$ of the eigenvalue problem (2.1) has a finite number of zeroes if $\lambda < \lambda_0$, and an infinite number of zeroes if $\lambda > \lambda_0$, with $\lambda_0$ given in formula (2.4). Thus, to finish the proof of Theorem 3, it is enough to prove that

$$\lambda_0 = 0 \text{ for } \alpha \geq 0 \text{ or } -1/a \leq \alpha < 0, \text{ and } \lambda_0 = \frac{k^2(1+a\alpha)^2}{2\alpha^2\sigma^2} > 0 \text{ if } \alpha < -1/a.$$

Indeed, using (2.3) and the properties of the functional $S$ given in the proof of Lemma 1, we obtain the following.

(i) If $\alpha = 0$ and when $x \to 0$ or $+\infty$, one has $S(x) \to \mp\infty$ and $\theta'(S(x))^2 \to +\infty$. When $x \to \exp(a)$, one has $\theta'(S(x))^2 \to 0$, hence $\lambda_0 = 0$.

(ii) If $\alpha > 0$, although $-\infty < S(0) < 0$ and $\theta'(S(x))^2 \to +\infty$ as $x \to +\infty$, the equation $x^\alpha = 1 + a\alpha$ has a solution $\bar{x} \in (0,+\infty)$ for which $\theta'(S(\bar{x}))^2 \to 0$, hence $\lambda_0 = 0$.

We split the case $\alpha < 0$ in three subcases, using each time (2.3). First remark that we have that $0 < S(+\infty) < +\infty$ and $\theta'(S(x))^2 \to +\infty$ as $x \to 0$.

(iii) If $1 + a\alpha > 0$ then equation $x^\alpha = 1 + a\alpha$ has a solution $\bar{x} \in (0,+\infty)$ for which $\theta'(S(\bar{x}))^2 \to 0$, hence $\lambda_0 = 0$.

(iv) If $1 + a\alpha = 0$, then $\inf_{x>0} \theta'(S(x))^2 = 0$, hence $\lambda_0 = 0$.

(v) If $1 + a\alpha < 0$, then equation $x^\alpha = 1 + a\alpha$ has no solution in $(0,+\infty)$, hence $\lambda_0 := \inf\limits_{x>0} \theta'(S(x))^2 > 0$ because, otherwise, there exists a sequence $x_n^\alpha \in (0,+\infty)$ such that $x_n^\alpha - (1 + a\alpha) \to 0$ when $n \to +\infty$, that is, $x_n^\alpha$ converges towards a solution of the equation $\theta' = 0$, and the latter fact would be a contradiction. Finally, remark that in this last subcase the critical value $\lambda_0$ is given by equation (2.2), by simply computing the inf value in (2.3) and then using formula (2.4). □

**Interpretation.** The simulations performed in [5] and [2] with the model (1.1) are, at microeconomic level, in agreement with the historical data on skewness, kurtosis, volatility smirk and spike prices provided that the parameters satisfy the condition $-1/a < \alpha < 0$. It worth explaining why model (1.1) exibits sufficiently many spikes (a main feature in electricty markets): the function $f_\alpha$ ($\alpha < 0$) approaches $-1/\alpha$ from below, i.e., a spike occurs as soon as the mean reverting process $\{Y_t\}_{t\in\mathbb{R}_+}$ approaches the barrier $-1/\alpha$. As the distribution of $Y_t$ is normal with mean $a$ and variance $\sigma^2/2k$ as $t \to \infty$, one would expect a sufficient number of spikes to occur if the mean level $a$ is within a few standard deviations of the barrier $-1/\alpha$, i.e.,

$$a + c\frac{\sigma}{\sqrt{2k}} = -\frac{1}{\alpha}, \quad \text{or} \quad 1 + a\alpha = -c\alpha\frac{\sigma}{\sqrt{2k}} > 0, \quad 1 \leq c \leq 3.$$

Also, condition $\lambda_0 > 0$ implies the existence of a gap on the spectrum to the left of 0, the latter being sufficient for the exponential decay of the diffusion $\rho$-mixing coefficients, see [7]. When $\lambda_0 = 0$, the diffusion is strongly dependent, that is, all $\rho$-mixing coefficients are identically equal to 1.

## 3. Options

We denote the appropriate discount factor by $r$ and, as the market is incomplete, in order to evaluate the price of a European option $g(X_T)$ with maturity $T > 0$ at time $t \in [0, T]$, we use the risk-minimizing hedge price with respect to the probability $P$ given by the conditional expectation

$$E^P[e^{-r(T-t)} g(X_T)|\mathcal{F}_t].$$

If $\mathcal{T}_{t,T}$ denotes the set of stopping times with respect to the Brownian filtration and taking values in $[t, T]$, then the price of an American option written on $X$ at time $t \in [0, T]$ is given by $\text{esssup}_{\tau\in\mathcal{T}_{t,T}}\ E^P[e^{-r(T-t)} g(X_\tau)|\mathcal{F}_t]$. It is unlikely that closed form expressions for such price functions will exist for general $\alpha$ despite the theoretical approach, using Malliavin derivatives (see Theorems 4.1 and 4.2 in [6]), and where $g$ is assumed to have linear growth at infinity. Hence we are led towards approximation techniques for options written on diffusions.

**Proposition 4.** *If $\alpha \leq 0$ and $g$ is uniformly continuous with sublinear growth (such as call options), then there exist finite time stochastic processes that approximate the price functions of European and American options written on $\{X_t\}_{t\in[0,T]}$.*

*Proof.* In the sequel $k$ denotes the integer part of $nt/T$. Consider the following discretization $X^{n,k} = X^{n,k-1} + T \cdot b(X^{n,k-1})/n + \sigma(X^{n,k-1})^{1-\alpha}\varepsilon_{n,k}$, where $\varepsilon_{n,k}$ are independent centered Gaussian random variables with variance $T/n$, or Bernoulli random variables taking values $\sqrt{T/n}$ and $-\sqrt{T/n}$ with probability $1/2$. We associate to $X^{n,k}$ a sequence of continuous time processes corresponding to the step processes $X_t^n$ such that $X_t^n = X^{n,k}$. The martingale difference array $\varepsilon_{n,k}$ satisfies the Lindeberg-Feller condition hence it converges to $W_t$ and, by [11], p. 167, the sequence of processes $\varepsilon_{n,1} + \varepsilon_{n,2} + \ldots + \varepsilon_{n,k}$ is uniformly tight. For $\alpha \leq 0$, the main result in [14] implies that $X^n$ converges in distribution to $X$ in the Skorohod topology. As $g$ is uniformly continuous and has sublinear growth, it follows that price function of European options written on $X^n$ converge to the price function of the European option written on $X$ in the Skorohod topology. The $X^n$ are in fact finite time stochastic processes, and therefore their Snell envelopes $U^n$ can be explicitly computed (see [12], p. 320) using a simple backwards recurrence formula. For $\alpha \leq 0$, Brémaud-Yor's $H$-hypothesis and Aldous' tightness criterion are satisfied, hence by Theorem 3.4 in [12], $U^n$ converges in distribution to $U$ in the Meyer-Zheng topology, where $U$ is the price of the American option written on $X$. Working under more restrictive hypothesis on $g$, both Skorohod and Meyer-Zheng topologies can be replaced by more familiar ones, see the details in [1]. $\square$

In spite of the algorithmic nature of the proof of Proposition 4, as shown in [1] and [12], there is poor control on the rate of convergence of the approximation error therein. Therefore, we adopt in the sequel the PDE approach, and search for a series representation

$$V(t,x) = \sum_{n=0}^{\infty} V_n(t,x) \text{ a.s.} \tag{3.1}$$

for the price of either European or American option $V(t, X_t)$ (by the strong Markov property of $X_t$, both prices are functions of $t$ and $X_t$ only). The following result shows that for $\alpha \leq 1/2$ the first coefficient $V_0$ of the above series is harmonic with respect to $x$, and indicates how to find recursively the other coefficients $V_n, n \geq 1$.

**Theorem 5.** *If $\alpha \leq 1/2$ and $g$ has at most polynomial growth at infinity (such as call or digital options), then $V$ is the unique solution of the following parabolic PDE on $[0,T) \times (0,+\infty)$:*

$$\frac{\partial V}{\partial t}(t,x) + \frac{\sigma^2}{2} x^{2-2\alpha} \frac{\partial^2 V}{\partial x^2}(t,x) + r\left[x \frac{\partial V}{\partial x}(t,x) - V(t,x)\right] = 0, \tag{3.2}$$

*that admits exponential growth in* $x$, *uniformly on the compact subintervals of* $[0, T)$. *In addition, there exists a series representation (3.1) whose coefficients* $V_n$ *satisfy:*

$$\frac{\partial^2 V_n}{\partial x^2}(t,x) = \begin{cases} 0 \quad \textit{for} \quad n = 0, \\ \dfrac{2x^{2\alpha-2}}{\sigma^2}\left[rV_{n-1}(t,x) - rx\dfrac{\partial V_{n-1}}{\partial x}(t,x) - \dfrac{\partial V_{n-1}}{\partial t}(t,x)\right] \quad \textit{for} \quad n \geq 1. \end{cases}$$

*Proof.* The first part is easily implied by the Feynman-Kac formula (see Theorem 5.7.6 and Remark 5.7.8 in [9]). We deduce that $V$ is the unique solution of PDE (3.2), together with the exponential growth in $x$, uniformly on the compact subintervals of $[0, T)$, as stated.

Inspired by a standard technique in large deviations theory, for $\varepsilon > 0$ we consider the diffusion process $X_t^\varepsilon$ whose dynamics are driven by

$$dX_t^\varepsilon = \frac{1}{\varepsilon} b(X_t^\varepsilon)dt + \frac{\sigma}{\sqrt{\varepsilon}}(X_t^\varepsilon)^{1-\alpha} dW_t.$$

The latter SDE is obtained from equation (1.2) by defining $X_t^\varepsilon := X_{t/\varepsilon}$ and noting that $W_{t/\varepsilon} = \varepsilon^{-1/2} W_t$ under $P$. Using the first part of Theorem 5, we obtain that the price function $V^\varepsilon(t,x)$ corresponding to the above-defined $X_t^\varepsilon$ is the unique solution of the following PDE on $[0,T) \times (0, +\infty)$:

$$\frac{\partial V^\varepsilon}{\partial t}(t,x) + \frac{\sigma^2}{2\varepsilon} x^{2-2\alpha} \frac{\partial^2 V^\varepsilon}{\partial x^2}(t,x) + r\left[x\frac{\partial V^\varepsilon}{\partial x}(t,x) - V^\varepsilon(t,x)\right] = 0. \tag{3.3}$$

We solve (3.3) by a series expansion $V^\varepsilon(t,x) = \sum_{n=0}^{\infty} V_n(t,x)\varepsilon^n$ a.s., and obtain

$$\left(\frac{\sigma^2}{2} x^{2-2\alpha} \frac{\partial^2 V_0}{\partial x^2}\right)\varepsilon^{-1} + \sum_{n=0}^{\infty}\left[\frac{\partial V_n}{\partial t} + \frac{\sigma^2}{2}x^{2-2\alpha}\frac{\partial^2 V_{n+1}}{\partial x^2} + rx\frac{\partial V_n}{\partial x} - rV_n\right]\varepsilon^n = 0. \tag{3.4}$$

To finish the proof, identify in (3.4) the coefficients of the $n$-th power of $\varepsilon$ for $n = -1, 0, 1, \ldots$ and remark that $\sum_{n=0}^{\infty} V_n(t,x)\varepsilon^n$ converges a.s. to $V^\varepsilon(t,x)$ for any $\varepsilon > 0$. □

**Remark.** The coefficients $V_n$ in Theorem 5 can be computed using standard approximating techniques, for instance using the Monte-Carlo Method (when $n = 0$) or the finite element method (when $n \geq 1$). These methods give a good control on the error approximation because equation (3.2) always has a data companion, namely a terminal data $V(T, x)$ in the case of a European option, or exercise/free boundary conditions for the price of an American option.

We thus obtained upper and lower bounds on the Value-at-Risk in the event of a price collapse with 1-in-20 chances of occurrence (for definitions and basic properties of Value-at-Risk, see e.g. [15]). Namely, for values of $1 + a\alpha$ ranging between 0.01 and 0.10, and during the spike price periods of time (1 to 10 day horizon, see Figures 1-4 in [2]), the losses of 0.12% or more will occur at least 5% of the time, while losses of 1.74% or more will occur at most 5% of the time.

The valuation of contingent claims requires the calculation of options' deltas. Using the method described in [15], Chapter 4, one obtains the so-called Grundy-Wiener-type upper and lower bounds of the European and American options' deltas written on Barlow's model when $g(x) = (x - K)^+$, where $K$ is the strike price. More precisely, for values of $1 + a\alpha$ ranging between 0.01 and 0.10, and the daily data in [2], Tables 1-3, the bounds of deltas for European option are: $[0.263; 0.384]$, and for American option: $[0.3384; 0.572]$.

While the Value-at-Risk bounds are quite good, the Grundy-Wiener bounds are not quite satisfactory, as one expects deltas at least equal to 0.5, like in the Black-Scholes world. There are two possible explanations for this fact. On one hand, to obtain more precise bounds for deltas, instead of using forward Euler approximations and maximum likelihood estimation, as in [2], one should bootstrap the original discrete time series and each time estimate nonparametrically the coefficients of the equation (1.2) using a local polynomial estimator. With these estimates one can construct the required confidence bands for the coefficients. On the other hand, the model itself might be subject to improvement, as the following test shows. The Pareto index $p$ measures how heavy the tails are in the slowly varying behavior of $x^p P[X_t > x]$ as $x \to \infty$. The popular Hill estimator is proved to be consistent for $1/p$ (see [13], Proposition 4.1) for a class of processes including Barlow's model. Applying the approximating method from [13], Section 5, for values of $1 + a\alpha$ ranging between 0.01 and 0.10, we obtained that the normalized Hill estimator for $p$ hovers from 1.43 to 2.12 in Barlow's model, *no matter the period of time we use.* This is in contradiction with the spike-producing data from Alberta and Northern California (see [2], Table 1 and Table 3) where, for certain periods of time, the value of $p$ obtained from the data is larger than 5.2! This facts suggest that, for a reasonable electricity model and associated price function, one should incorporate short and long term effects in order to explain skewness and excess kurtosis; hence some jump components should be "pasted" to $X_t$, eventually based on discontinuous coefficients.

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