International Journal of Pure and Applied Mathematics

Volume 21 No. 3 2005, 307-314

STARLIKENESS, CONVEXITY, CLOSE-TO-CONVEXITY, AND QUASI-CONVEXITY OF CERTAIN ANALYTIC FUNCTIONS

H. Irmak[1 §], K. Piejko[2]

[1]Department of Mathematical Education
Faculty of Education
Başkent University
Bağlıca Campus, Ankara, 06530, TURKEY
e-mail: hisimya@baskent.edu.tr

[2]Department of Mathematics
Rzeszów University of Technology
Ul. W. Pola 2, Rzeszów, 35-959, POLAND
e-mail: piejko@prz.rzeszow.pl

Abstract: The aim of the present investigation is to state a theorem including some new conditions for starlikeness, convexity, close-to-convexity, and quasi-convexity of certain normalized functions which are analytic and univalent in the unit open disc $\mathcal{U}$. Some consequences of our results are also given.

AMS Subject Classification: 30C45, 30A10
Key Words: normalized analytic functions, univalent functions, convex functions, starlike functions, close-to-convex functions, quasi-convex functions, inequalities, Jack's Lemma, Nunokawa's Lemma

1. Introduction and Definitions

Let $\mathcal{A}_n$ denote the class of *normalized* functions $f(z)$ of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \quad (n \in \mathbf{N} \equiv \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are *analytic and univalent* in the *open unit disc* $\mathcal{U} = \{z \in \mathbf{C} \ : \ |z| < 1\}$.

Received: April 25, 2005

§Correspondence author

Let also $\mathcal{S}_n^*(\alpha)$, $\mathcal{K}_n(\alpha)$, $\mathcal{C}_n(\alpha,\beta)$ and $\mathcal{C}_n^*(\alpha,\beta)$ denote the subclasses of the class $\mathcal{A}_n$ consisting of functions which are, respectively, *starlike of order* α, *convex of order* α, *close-to-convex of order* β *and type* α, and *quasi-convex of order* β *and type* α in $\mathcal{U}$, where $0 \le \alpha < 1$ and $0 \le \beta < 1$.

It is obvious that

$$\mathcal{S}_n^*(\alpha) := \left\{ f \ : \ f \in \mathcal{A}_n \ \text{ and } \ \Re e\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\},$$

$$\mathcal{K}_n(\alpha) := \left\{ f \ : \ f \in \mathcal{A}_n \ \text{ and } \ \Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\},$$

$$\mathcal{C}_n(\alpha,\beta) := \left\{ f \ : \ f \in \mathcal{A}_n \ , \ \ \Re e\left(\frac{zf'(z)}{g(z)}\right) > \beta \ \text{ and } \ g \in \mathcal{S}_n^*(\alpha) \right\},$$

and

$$\mathcal{C}_n^*(\alpha,\beta) := \left\{ f \ : \ f \in \mathcal{A}_n \ , \ \ \Re e\left(\frac{[zf'(z)]'}{g'(z)}\right) > \beta \ \text{ and } \ g \in \mathcal{K}_n(\alpha) \right\},$$

where for some $0 \le \alpha < 1$ and $0 \le \beta < 1$, and for all $z \in \mathcal{U}$.

From the above definitions, it is clear that:

$$f(z) \in \mathcal{K}_n(\alpha) \quad \Leftrightarrow \quad zf'(z) \in \mathcal{S}_n^*(\alpha)$$

and

$$f(z) \in \mathcal{C}_n^*(\alpha,\beta) \quad \Leftrightarrow \quad zf'(z) \in \mathcal{C}_n(\alpha,\beta),$$

where $f(z) \in \mathcal{A}_n$. For the details of above definitions, see, [1], [2], and also [6] and [8].

In the present paper, we establish two novel condition which include starlike of order α, convex of order α, close-to-convex of order α, and quasi-convex of order β and type α in $\mathcal{U}$, where $0 \le \alpha < 1$ and $0 \le \beta < 1$. For their proofs we used the results of Jack and Nunokawa (Lemma 1.1 and Lemma 1.2 below) which are popularly known as Jack's Lemma and Nonokawa's Lemma in the literature.

Lemma 1.1. (see [5]) *Let* $w(z)$ *be non-constant and analytic in* $\mathcal{U}$ *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r$ $(0 < r < 1)$ *at the point* z_0, *then* $z_0 w'(z_0) = cw(z_0)$, *where* $c \ge 1$.

Lemma 1.2. (see [7]) *Let* $p(z)$ *be an analytic function in* $\mathcal{U}$ *with* $p(0) = 1$. *If there exists a point* $z_0 \in \mathcal{U}$ *such that*

$$\Re e\,\{p(z)\} > 0 \ (|z| < |z_0|), \quad \Re e\,\{p(z_0)\} = 0 \ \ \textit{and} \ \ p(z_0) \ne 0,$$

then

$$p(z_0) = ia \quad \text{and} \quad \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a + \frac{1}{a}\right),$$

where $a \neq 0$ *and* $c \geq 1$.

2. The Main Results

Using Lemma 1.1 and Lemma 1.2 we now prove the following result (Theorem below).

Theorem. *Let* $z \in \mathcal{U}$, $0 \leq \lambda \leq 1$, $f(z) \in \mathcal{A}_n$, *and* $g(z) \in \mathcal{A}_n$ *with* $f(z) \neq g(z)$, *and also let the functions* $\mathcal{F}(z)$ *and* $\mathcal{G}(z)$ *be defined by*

$$\mathcal{F}(z) := zf'(z) + \lambda z^2 f''(z) \tag{2.1}$$

and

$$\mathcal{G}(z) := (1-\lambda)g(z) + \lambda z g'(z). \tag{2.2}$$

(i) If

$$\Re e\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right) - \Re e\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) < \frac{1-\alpha}{2(1+\alpha)}, \tag{2.3}$$

then

$$\Re e\left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)}\right) > \frac{1+\alpha}{2} \quad (0 \leq \alpha < 1). \tag{2.4}$$

(ii) If

$$\Re e\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) > \Re e\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right), \tag{2.5}$$

then

$$\Re e\left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)}\right) > \alpha \quad (0 \leq \alpha < 1). \tag{2.6}$$

Proof. First we prove (i). From the definitions (1.1), (2.1) and (2.2) we can easily see that:

$$\frac{\mathcal{F}(z)}{\mathcal{G}(z)} = \frac{z + \sum\limits_{k=n+1}^{\infty} b_k z^k}{z + \sum\limits_{k=n+1}^{\infty} c_k z^k} = 1 + d_1 z + d_2 z^2 + \ldots \qquad (z \in \mathcal{U}),$$

and if we define a function $w(z)$ by

$$\frac{\mathcal{F}(z)}{\mathcal{G}(z)} = \frac{1+\alpha w(z)}{1+w(z)} \quad (z \in \mathcal{U};\ 0 \leq \alpha < 1), \tag{2.7}$$

then, clearly, $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$. We can also find from (2.7) that

$$z\left(\frac{\mathcal{F}'(z)}{\mathcal{F}(z)}-\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right)=\frac{\alpha z w'(z)}{1+\alpha w(z)}-\frac{zw'(z)}{1+w(z)} \quad (z\in\mathcal{U}). \tag{2.8}$$

Assume now that there exists a point $z_0\in\mathcal{U}$ such that

$$|w(z_0)|=1, \quad \text{and} \quad |w(z)|<1 \quad \text{when} \quad |z|<|z_0| \quad (z\in\mathcal{U}).$$

Then, applying Lemma 1.1, we have

$$z_0 w'(z_0)=cw(z_0) \quad \left(c\geq 1;\ w(z_0)=e^{i\varphi};\ 0\leq\varphi<2\pi\right). \tag{2.9}$$

So, (2.8) and (2.9) yield that

$$\begin{aligned}\Re e\left\{z\left(\frac{\mathcal{F}'(z)}{\mathcal{F}(z)}-\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\Bigg|_{z=z_0}\right\}&=\Re e\left\{\frac{\alpha z_0 w'(z_0)}{1+\alpha w(z_0)}-\frac{z_0 w'(z_0)}{1+w(z_0)}\right\}\\ &=c\ \Re e\left(\frac{\alpha e^{i\varphi}}{1+\alpha e^{i\varphi}}-\frac{e^{i\varphi}}{1+e^{i\varphi}}\right)\\ &\leq\frac{\alpha-1}{2(1+\alpha)} \quad (0\leq\alpha<1),\end{aligned}$$

which is a contradiction to the condition (2.3). Therefore $|w(z)|<1$ for all $z\in\mathcal{U}$. Hence, the definition (2.7) yields

$$\left|\frac{1-\frac{\mathcal{F}(z)}{\mathcal{G}(z)}}{\frac{\mathcal{F}(z)}{\mathcal{G}(z)}-\alpha}\right|=|w(z)|<1 \quad (0\leq\alpha<1; z\in\mathcal{U}),$$

which implies the inequality (2.4). This completes the proof of (i) in Theorem.

For the proof of (ii), we again define a new function $p(z)$ by

$$\frac{\mathcal{F}(z)}{\mathcal{G}(z)}=\alpha+(1-\alpha)p(z) \quad (z\in\mathcal{U};\ 0\leq\alpha<1), \tag{2.10}$$

where $p(z)$ is analytic in $\mathcal{U}$ with $p(0)=1$. Then, we easily find from (2.10) that

$$z\left(\frac{\mathcal{F}'(z)}{\mathcal{F}(z)}-\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right)=\frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)} \quad (z\in\mathcal{U};\ 0\leq\alpha<1). \tag{2.11}$$

Suppose now that there exists a point $z_0\in\mathcal{U}$ such that

$$\Re e\,\{p(z)\}>0 \quad (|z|<|z_0|), \quad \Re e\,\{p(z_0)\}=0, \quad \text{and} \quad p(z_0)\neq 0 \quad (z\in\mathcal{U}).$$

Then, by using Lemma 1.2, we have

$$p(z_0) = ia \quad \text{and} \quad \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a + \frac{1}{a}\right) \quad (a \neq 0;\ c \geq 1). \tag{2.12}$$

Thus, we have from (2.12) and (2.11) that

$$\Re e\left\{ z\left(\frac{\mathcal{F}'(z)}{\mathcal{F}(z)} - \frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\Bigg|_{z=z_0}\right\} = \Re e\left\{\frac{(1-\alpha)z_0 p'(z_0)}{p(z_0)} \cdot \frac{p(z_0)}{\alpha + (1-\alpha)p(z_0)}\right\}$$

$$= -\frac{c\alpha(1-\alpha)(1+a^2)}{2[\alpha^2 + a^2(1-\alpha)^2]} \leq 0 \quad (0 \leq \alpha < 1),$$

which contradicts the condition (2.5). Hence $\Re e\{p(z)\} > 0$ for all $z \in \mathcal{U}$ and the equality (2.10) implies the condition (2.6). Therefore, the proof of Theorem is completed. □

Since for $\lambda = 0$ the equations (2.1) and (2.2) imply that $\mathcal{F}(z) = zf'(z)$ and $\mathcal{G}(z) = g(z)$, respectively, then (by setting $\lambda = 0$ in Theorem) we obtain the following result.

Corollary 2.1. *Let $f(z) \in \mathcal{A}_n$ and $g(z) \in \mathcal{A}_n$ with $f(z) \neq g(z)$. If*

$$\Re e\left\{ z\left(\frac{g'(z)}{g(z)} - \frac{f''(z)}{f'(z)}\right)\right\} < \frac{3+\alpha}{2(1+\alpha)},$$

then

$$\Re e\left(\frac{zf'(z)}{g(z)}\right) > \frac{1+\alpha}{2} \quad (z \in \mathcal{U},\ 0 \leq \alpha < 1).$$

Moreover, if

$$\Re e\left\{ z\left(\frac{f''(z)}{f'(z)} - \frac{g'(z)}{g(z)}\right)\right\} > -1,$$

then

$$\Re e\left(\frac{zf'(z)}{g(z)}\right) > \alpha \quad (z \in \mathcal{U};\ 0 \leq \alpha < 1).$$

Since for $\lambda = 1$ the equations (2.1) and (2.2) imply that $\mathcal{F}(z) = z\left[zf'(z)\right]'$ and $\mathcal{G}(z) = zg'(z)$, respectively, then (by setting $\lambda = 1$ in Theorem) we find the following result.

Corollary 2.2. *Let $f(z) \in \mathcal{A}_n$ and $g(z) \in \mathcal{A}_n$ with $f(z) \neq g(z)$. If*

$$\Re e\left\{ z\left(\frac{g''(z)}{g'(z)} - \frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)}\right)\right\} < \frac{1-\alpha}{2(1+\alpha)},$$

then

$$\Re e\left(\frac{f'(z)+zf''(z)}{g'(z)}\right) > \frac{1+\alpha}{2} \quad (z\in\mathcal{U};\ 0\leq\alpha<1).$$

Moreover, if

$$\Re e\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{g''(z)}{g'(z)}\right)\right\} > 0,$$

then

$$\Re e\left(\frac{f'(z)+zf''(z)}{g'(z)}\right) > \alpha \quad (z\in\mathcal{U};\ 0\leq\alpha<1).$$

If we put $g(z) := f(z)$ in Corollary 2.1 and Corollary 2.2, respectively, then we receive the following results.

Corollary 2.3. *Let* $f(z)\in\mathcal{A}_n$, $z\in\mathcal{U}$, *and* $0\leq\alpha<1$. *If the function* $f(z)$ *satisfies the condition:*

$$\Re e\left\{z\left(\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)}\right)\right\} < \frac{3+\alpha}{2(1+\alpha)},$$

then

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \frac{1+\alpha}{2}, \quad \text{i.e.,} \quad f(z)\in\mathcal{S}_n^*\left(\frac{1+\alpha}{2}\right).$$

Moreover, if it satisfies the condition:

$$\Re e\left\{z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\right\} > -1,$$

then

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad \text{i.e.,} \quad f\in\mathcal{S}_n^*(\alpha).$$

Corollary 2.4. *Let* $f(z)\in\mathcal{A}_n$, $z\in\mathcal{U}$, *and* $0\leq\alpha<1$. *If the function* $f(z)$ *satisfies the condition:*

$$\Re e\left\{z\left(\frac{f''(z)}{f'(z)}-\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}\right)\right\} < \frac{1-\alpha}{2(1+\alpha)},$$

then

$$\Re e\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{1+\alpha}{2}, \quad \text{i.e.,} \quad f(z)\in\mathcal{K}_n\left(\frac{1+\alpha}{2}\right).$$

Moreover, if it satisfies the condition:

$$\Re e\left\{z\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{f''(z)}{f'(z)}\right)\right\} > 0,$$

then

$$\Re e\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha, \quad i.e., \quad f(z)\in\mathcal{K}_n(\alpha).$$

If we choose the function $g(z)\in\mathcal{A}_n$ in Corollary 2.1 as $g(z)$ in the class $\mathcal{S}_n^*(\gamma)$ $(0\le\gamma<1)$, then we get the following results.

Corollary 2.5. *Let $f(z)\in\mathcal{A}_n$, $g(z)\in\mathcal{S}_n^*(\gamma)$, $0\le\gamma<1$ and $z\in\mathcal{U}$. If*

$$\Re e\left\{z\left(\frac{g'(z)}{g(z)}-\frac{f''(z)}{f'(z)}\right)\right\}<\frac{3+\alpha}{2(1+\alpha)},$$

then $f(z)\in\mathcal{C}_n\left(\frac{1+\alpha}{2},\gamma\right)$. Moreover, if

$$\Re e\left\{z\left(\frac{f''(z)}{f'(z)}-\frac{g'(z)}{g(z)}\right)\right\}>-1,$$

then $f(z)\in\mathcal{C}_n(\alpha,\gamma)$.

If we choose the function $g(z)\in\mathcal{A}_n$ in Corollary 2.2 as $g(z)$ in the class $\mathcal{K}_n(\gamma)$ $(0\le\gamma<1)$, then we arrive at the following results.

Corollary 2.6. *Let $f(z)\in\mathcal{A}_n$, $g(z)\in\mathcal{K}_n(\gamma)$, $0\le\gamma<1$, $0\le\alpha<1$ and $z\in\mathcal{U}$. If*

$$\Re e\left\{z\left(\frac{g''(z)}{g'(z)}-\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}\right)\right\}<\frac{1-\alpha}{2(1+\alpha)},$$

then $f(z)\in\mathcal{C}_n^*\left(\frac{1+\alpha}{2},\gamma\right)$. Moreover, if

$$\Re e\left\{z\left(\frac{zf''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{g''(z)}{g'(z)}\right)\right\}>0,$$

then $f(z)\in\mathcal{C}_n^*(\alpha,\gamma)$.

Remark. Corollary 2.3 and Corollary 2.4 are comparable with the certain results given earlier by Irmak et al [3] and [4].

2. Acknowledgements

This work has been carried out by the help of four-month financial support (June-September, 2004) from the TÜBİTAK (The Scientific and Technical Research Council of TURKEY) which is given to the first author, during the scientific research at the University of Rzeszów and Rzeszów University of Technology in Poland. This present investigation was also supported by NATO and Başkent University (Ankara,TURKEY). The first author would also like to acknowledge Professor Mehmet Haberal, Rector of Başkent University, who generously supports scientific researches in all aspects. I would like to extend my thanks to Professor J. Stankiewicz and Professor J. Dziok for their kind invitation to Poland and their invaluable support for this research.

References

[1] P.L. Duren, *Univalent Functions,* Grundlehren der Mathematishen Wissenschaften, **259**, Springer-Verlag (1983).

[2] A.W. Goodman, *Univalent Functions.* Volume **I** and Volume **II**, Polygonal Publishing Company (1983).

[3] H. Irmak, Ö. F. Çetin, Some theorems involving inequalities on p-valent functions, *Turk. J. Math.*, **23** (1999), 453-459.

[4] H. Irmak, G. Tinaztepe, Y.C. Kim, J.H. Choi, Certain classes and inequalities involving fractional calculus and multivalent functions, *Fract. Calc. Appl. Anal.*, **5**, No. 3 (2002), 227-274.

[5] I.S. Jack, Functions starlike and convex of order α, *J. London Math. Soc.*, **3** (1971), 469-474.

[6] K.I. Noor, On quasi-convex functions and related topics, *Inter. J. Math. and Math. Sci.*, **10** (1987), 241-258.

[7] M. Nunokawa, On the properties of non-caratheodory functions, *Proc. Japan. Acad. Ser. A Math. Sci.*, **68** (1992), 152-153.

[8] H.M. Srivastava, S. Owa, Ed-s., *Current Topics in Analytic Functions Theory,* World Scientific Publ. Comp. (1992).