**International Journal of Pure and Applied Mathematics**

**Volume 21 No. 3 2005, 315-318**

# UPPER AND LOWER BOUNDS FOR THE SPECTRAL RADIUS OF THE OPERATOR $A_\rho(\alpha)$

Julio Alcántara-Bode
Instituto de Matemáticas y Ciencias Afines (IMCA)
Pontificia Universidad Católica del Perú (PUCP)
Dean Saavedra 253, Lima, 32, PERU
e-mail: jalcant@pucp.edu.pe

**Abstract:** We give upper and lower bounds for the spectral radius of a family of compact operators used to study the Riemann Hypothesis.

**AMS Subject Classification:** 45M20, 47A12, 47B60, 47B65, 11M26
**Key Words:** positive operators, spectral radius, Riemann Hypothesis

## 1. Introduction

In a previous work [1] we have studied the integral operator on $L^2(0,1)$,

$$[A_\rho(\alpha)f](\theta) = \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) f(x)dx,$$

where $0 < \alpha < 1$ and $\rho(x) = x - [x]$ is the fractionary part function. The main results of this work are the following:

(i) $A_\rho(\alpha)$ is Hilbert-Schmidt, but neither nuclear nor normal.

(ii) $\lambda \neq 0$ is an eigenvalue of $A_\rho(\alpha)$ if and only if $T_\alpha(\lambda^{-1}) = 0$, where

$$T_\alpha(\mu) = 1 - \alpha\mu + \sum_{r=1}^{\infty} \frac{(-1)^{r+1}\alpha^{(r+1)(r+2)/2}}{(r+1)!(r+1)} \prod_{\ell=1}^{r} \zeta(\ell+1)\mu^{r+1}$$

is an entire function of order zero. Moreover each non-zero eigenvalue $\lambda$ has geometric multiplicity one and associated eigenfunction $\psi_{\lambda^{-1}}(x) = \frac{x}{\lambda}T'_\alpha\left(\frac{x}{\lambda}\right)$; explicit formulae are also given for the generalized eigenfunctions of non-zero eigenvalues that have algebraic multiplicity greater than one.

Received: April 27, 2005 

(iii) $\dim\ker A_\rho(\alpha) = \infty$ and $\ker A_\rho(\alpha)^* = \{0\}$ if $0 < \alpha < 1$.

(iv) If $D^*(\mu)$ is the renormalized or modified Fredholm determinant of $I - \mu A_\rho(\alpha)$, then

$$D^*_\alpha(\mu) = e^{\alpha\mu} T_\alpha(\mu)\,, \quad \forall \mu \in \mathbb{C}\,.$$

(v) If $\{\lambda_n(\alpha)\}_{n\geq 1}$ is the sequence of non-zero eigenvalues of $A_\rho(\alpha)$, where the ordering is such that $|\lambda_n(\alpha)| \geq |\lambda_{n+1}(\alpha)| \ \ \forall n \in \mathbb{N}$, and each eigenvalue is repeated according to his algebraic multiplicity, then the sequence is infinite, $\sum\limits_{n=1}^{\infty} |\lambda_n(\alpha)|^r < \infty$, $\forall r > 0$, the first eigenvalue $\lambda_1(\alpha)$ is positive, has algebraic multiplicity one and $\lambda_1(\alpha) > |\lambda_j(\alpha)|\, \forall j > 1$.

(vi) The set of eigenvectors and generalized eigenvectors of $A_\rho(\alpha)$ associated to its non-zero eigenvalues is total in $L^2(0,1)$, but it is not part of a Markushevich, basis in $L^2(0,1)$ and therefore it is not a Schauder basis in $L^2(0,1)$.

(vii) If $0 < \alpha < 1$, the Riemann Hypothesis holds if and only if $\overline{R(A_\rho(\alpha)^*)} \supset L^2(0,\alpha)$.

## 2. Proof of Main Result

The purpose of this note is to prove that

$$\min\left(\frac{\alpha}{2}, \frac{1}{\sqrt{2}+1}\right) \leq r(A_\rho(\alpha)) \leq \min\left(\alpha, \frac{1}{2}\right)\,, \quad \forall \alpha \in [0,1]\,,$$

where $r(A_\rho(\alpha)) = \lim \|A_\rho(\alpha)^n\|^{1/n}$ is the spectral radius of $A_\rho(\alpha)$. From general theory we know that

$$r(A_\rho(\alpha)) = \max_{\lambda\in\sigma(A_\rho(\alpha))} |\lambda|\,,$$

where $\sigma(A_\rho(\alpha))$ is the spectrum of $A_\rho(\alpha)$. But from

$$\max_{\lambda\in\sigma(A_\rho(\alpha))} |\lambda| = \lambda_1(\alpha),$$

we get that

$$r(A_\rho(\alpha)) = \lambda_1(\alpha)\,.$$

Now $\lambda_1(\alpha)$ is also an eigenvalue of $A_\rho(\alpha)^*$ and the corresponding eigenfunction $\chi_1$, that can be taken as real, is continuous and does not change sign (this last property follows from the theorem of Krein-Rutman [3]). We will assume that $\chi_1 \geq 0$. Since $\chi_1(1) \neq 0$, it is not difficult to show that $\min_{x\in[0,1]} \chi_1(x) > 0$.

From the eigenvalue equation

$$\lambda_1(\alpha)\chi_1(x) = \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) \chi_1(\theta) d\theta \,,$$

it follows that

$$\lambda_1(\alpha)\chi_1(x) \leq \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) d\theta \max_{\theta\in[0,1]} \chi_1(\theta) \tag{1}$$

and

$$\lambda_1(\alpha)\chi_1(x) \geq \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) d\theta \min_{\theta\in[0,1]} \chi_1(\theta) \,. \tag{2}$$

Now if $\frac{\alpha\theta}{x} \notin \mathbb{Z}$ it holds that

$$\rho\left(\frac{\alpha\theta}{x}\right) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin\left(2\pi\frac{n\alpha\theta}{x}\right)}{n\pi} \,. \tag{3}$$

The series in (3) is boundedly convergent, and therefore term by term integration is permissible, to prove that

$$\int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) d\theta = \frac{1}{2} - \frac{x}{12\alpha} + \frac{x}{2\pi^2\alpha} \sum_{n=1}^{\infty} \frac{\cos\left(2\pi\frac{n\alpha}{x}\right)}{n^2} \,.$$

But by T12.19 in [2]

$$\sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{k^2} = \pi^2\left(x^2 - x + \frac{1}{6}\right) \,, \quad x \in [0,1] \,,$$

therefore we finally get that

$$\int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) d\theta = \frac{1}{2} - \frac{x}{2\alpha}\rho\left(\frac{\alpha}{x}\right)\left(1 - \rho\left(\frac{\alpha}{x}\right)\right) \,. \tag{4}$$

The function $h_\alpha(x) = \frac{x}{\alpha}\rho\left(\frac{\alpha}{x}\right)\left(1-\rho\left(\frac{\alpha}{x}\right)\right)$ is continuous and

$$\max_{x\in[0,1]} h_\alpha(x) = \max\left(\frac{1}{(\sqrt{2}+1)^2}, 1-\alpha\right) \,, \quad \min_{x\in[0,1]} h_\alpha(x) = 0 \,. \tag{5}$$

From (1), (2), (3), (4) and (5) we get that

$$\min\left(\frac{\alpha}{2},\frac{1}{\sqrt{2}+1}\right) \leq \lambda_1(\alpha) \leq \frac{1}{2}\,. \tag{6}$$

We now try to improve on the upper bound in (6) by looking at the eigenvalue equation

$$\lambda_1(\alpha)\theta\psi_1(\theta) = \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) x\psi_1(x)dx\,, \tag{7}$$

where $\psi_1(\theta) = -T'_\alpha\left(\frac{\theta}{\lambda_1(\alpha)}\right) \geq 0$, $\theta \in [0,1]$.

From (7) we get that

$$\lambda_1(\alpha)\,\theta\psi_1(\theta) \leq \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) xdx \max_{x\in[0,1]} \psi_1(x)\,,$$

or

$$\lambda_1(\alpha)\,\theta\psi_1(\theta) \leq \alpha\theta\left(1-\frac{\zeta(2)}{2}\alpha\,\theta\right) \max_{x\in[0,1]} \psi_1(x)\,, \tag{8}$$

which finally gives

$$\lambda_1(\alpha) \leq \alpha\,. \tag{9}$$

From (6) and (9) we obtain

$$\min\left(\frac{\alpha}{2},\frac{1}{\sqrt{2}+1}\right) \leq \lambda_1(\alpha) \leq \min\left(\alpha,\frac{1}{2}\right)\,, \quad \forall\alpha\in[0,1].$$

## References

[1] J. Alcántara-Bode, A completeness problem related to the Riemann Hypothesis, *Journal of Integral Equation and Operator Theory*, To Appear.

[2] T.M. Apostol, *Introducción a la Teoría Analítica de Números*, Reverté, Barcelona (1980).

[3] M. Zerner, Quelques propiétés spectrales des operateurs positifs, *Journal of Functional Analysis*, **72** (1987), 381-417.