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A COMBINED METHOD FOR SOLVING LAPLACE'S BOUNDARY VALUE PROBLEM WITH SINGULARITIES

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Abstract: A combined Block-Grid Method (see Dosiyev [2], [3], [4], Dosiyev and Cival [5]) for the solution of the Dirichlet problem on polygons, when a boundary function on some sides is given from $C_{1,1}$ is analized. The obtained uniform estimate for the error of the approximate solution is of order $O\left(h^2\left|\ln h\right|+1\right)$, whereas it is of order $O(h^2(|\ln h|+1)/r_j^{p-1/\alpha_j})$ for the errors of $p-$order derivatives $(p=1,2,...)$ in a finite neighbourhood of vertices; here h is the mesh step, r_j is the distance from the current point to the vertex in question, $\alpha_j\pi$ is the value of the angle.

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1. Introduction

It is well known that the use of classical finite difference or finite element methods to solve the elliptic boundary value problems with singularities becomes ineffective. A special construction is usually needed for the numerical scheme near the singularities in such a way that the order of convergence is the same

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as in the case of a smooth solution. Among many approaches to solve this problem, a special emphasis has been placed on the construction of combined methods, in which differential properties of the solution in different parts of the domain are used (see Li [1], Dosiyev [2], [3], [4], Dosiyev and Cival [5], Lucas and Oh [6], and Wu and Han [7], and references therein).

In Dosiyev [2], [3], [4], [5] a new combined difference-analytical method called the Block-Grid Method (BGM) in solving the Laplace equation on polygons is introduced. This method is a combination of the exponentially convergent Block Method (see Volkov [8], [9]) in "singular" part, and the finite difference method in "nonsingular" part of the polygon. The uniform estimate of the error of the BGM is of order $O(h^k)$ (h is the mesh step), when the given boundary function on some sides of polygon is given from the Hölder classes $C_{k,\lambda}$, $0 < \lambda < 1$ (see Dosiyev [2], [3], [4] for $k = 6$, and Dosiyev and Cival [5] for $k = 2$).

In this paper, the error of the BGM for the solution of the Dirichlet problem on arbitrary polygons is estimated when the boundary function is from $C_{1,1}$, i.e., has first derivative, which satisfies a Lipschitz condition. The uniform estimate for the error of the approximate solution is of order $O(h^2(|\ln h| + 1))$, whereas it is of order $O(h^2(|\ln h| + 1)/r_j^{p-1/\alpha_j})$ for the errors of $p-$order derivatives ($p = 1, 2, ...$) in a finite neighborhood of the vertices; r_j is the distance from the current point to the vertex in question. The system of finite difference equations on the union of all rectangles may be solved by the alternating method of Schwarz with the number of iterations $O(\ln \varepsilon^{-1})$, where ε is the prescribed accuracy, by solving standard 5-point difference equations of Laplace on rectangular domain at each iteration.

2. Boundary Value Problem on Polygons

Let G be an open simply connected polygon, γ_j, $j = 1(1)N$, be its sides, including the ends, enumerated counterclockwise, $\gamma = \gamma_1 \cup ... \cup \gamma_N$ be the boundary of G, $\alpha_j\pi$, $0 < \alpha_j \leq 2$, be the interior angle formed by the sides γ_{j-1} and $\gamma_j, (\gamma_0 = \gamma_N)$, $A_j = \gamma_{j-1} \cap \gamma_j$ be the vertex of the $j-$th angle, r_j, θ_j be a polar system of coordinates with pole in A_j and the angle θ_j taken counterclockwise from the side γ_j.

We consider the boundary value problem

$$\Delta u = 0 \text{ on } G, \quad u = \varphi_j(s) \text{ on } \gamma_j, \quad 1 \leq j \leq N, \tag{1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j is given function of the arc length s taken along

γ. At the vertices $A_j, (s = s_j)$ for $\alpha_j = 1/2$ the continuity condition $\varphi_{j-1} = \varphi_j$ is fulfilled, and for $\alpha_j \neq 1/2$ the values of φ_{j-1} and φ_j at A_j might be different. We require that when $\alpha_j \neq 1/2$ the boundary functions on γ_{j-1} and on γ_j be given as algebraic polynomials of s, and on the remainder sides γ_ν, $1 < \nu < N$, of the polygon G

$$\varphi_\nu \in C_{1,1}(\gamma_\nu), \tag{2}$$

i.e., φ_ν has the first derivative, which satisfies a Lipschitz condition.

We represent the given boundary functions (algebraic polynomials) on γ_{j-1} and γ_j for $\alpha_j \neq 1/2$ in the form

$$\sum_{k=0}^{\tau_{j-1}} a_{jk}^0 r_j^k \quad \text{and} \quad \sum_{k=0}^{\tau_j} b_{jk}^0 r_j^k, \tag{3}$$

respectively, where a_{jk}^0 and b_{jk}^0 are numerical coefficients and τ_{j-1} and τ_j are the degrees of those polynomials.

Let E be the set $j, (1 \leq j \leq N)$ for which $\alpha_j \neq 1/2$. In the neighborhood of A_j, $j \in E$ we construct two fixed block-sectors $T_j^i = T_j(r_{ji}) \subset G$, $i = 1, 2$, where $0 < r_{j2} < r_{j1} < \min\{s_{j+1} - s_j, s_j - s_{j-1}\}$, $T_j(r) = \{(r_j, \theta_j) : 0 < r_j < r,$ $0 < \theta_j < \alpha_j \pi\}$.

On the closed sector $\overline{T_j^1}$, $j \in E$ we consider a function $Q_j(r_j, \theta_j)$ with the next properties:

(a) $Q_j(r_j, \theta_j)$ is harmonic and bounded on the open sector T_j^1;

(b) continuous on $\overline{T_j^1}$ everywhere, except for the point A_j (the vertex of the sector) when $a_{j0}^0 \neq b_{j0}^0$;

(c) continuously differentiable on $\overline{T_j^1} \setminus A_j$ and satisfies the boundary conditions in (1) on $\gamma_{j-1} \cap \overline{T_j^1}$ and $\gamma_j \cap \overline{T_j^1}$, $j \in E$.

For definiteness we assume that $Q_j(r_j, \theta_j)$ with the above properties has the form (3.1) given by Volkov [10].

Remark 1. We formally set the value of $Q_j(r_j, \theta_j)$ and the solution u of problem (1) at the vertex A_j equal to $(a_{j0}^0 + b_{j0}^0)/2$.

Let

$$R_j(r, \theta, \eta) = \frac{1}{\alpha_j} \sum_{k=0}^{1} (-1)^k R\left(\left(\frac{r}{r_{j2}}\right)^{\frac{1}{\alpha_j}}, \frac{\theta}{\alpha_j}, (-1)^k \frac{\eta}{\alpha_j}\right), \ j \in E,$$

where

$$R(r, \theta, \eta) = \frac{1 - r^2}{2\pi(1 - 2r\cos(\theta - \eta) + r^2)}, \tag{4}$$

is the kernel of the Poisson integral for a unit circle.

Lemma 2. (see Volkov [9]) *The solution u of the boundary value problem (1) can be represented on $\overline{T_j^2} \setminus V_j$, $j \in E$ in the form*

$$u(r_j, \theta_j) = Q_j(r_j, \theta_j) + \int_0^{\alpha_j \pi} R_j(r_j, \theta_j, \eta)(u(r_{j2}, \eta) - Q_j(r_{j2}, \eta))d\eta, \quad (5)$$

where V_j is the curvilinear part of the boundary of T_j^2.

3. Description of the Block-Grid Method

In addition to the sectors T_j^1 and T_j^2 (see Section 2) in the neighborhood of each vertex A_j, $j \in E$ of the polygon G we construct two more sectors T_j^3 and T_j^4, where $0 < r_{j4} < r_{j3} < r_{j2}$, $r_{j3} = (r_{j2} + r_{j4})/2$ and $T_k^3 \cap T_l^3 = \emptyset$, $k \neq l$, $k, l \in E$, and let $G_T = G \setminus (\cup_{j \in E} T_j^4)$.

Let $\Pi_k \subset G_T$, $k = 1(1)M$, $(M < \infty)$ be certain fixed open rectangles with arbitrary orientation, generally speaking, with sides a_{1k} and a_{2k}, a_{1k}/a_{2k} being rational and $G = (\cup_{k=1}^M \Pi_k) \cup (\cup_{j \in E} T_j^3)$. Let η_k be the boundary of the rectangle Π_k and V_j be the curvilinear part of the boundary of the sector T_j^2, and $t_j = (\cup_{k=1}^M \eta_k) \cap \overline{T_j^3}$. The following general requirement is imposed on the arrangement of the rectangles Π_k, $k = 1(1)M$ and sectors T_j^2, $j \in E$: any point P lying on $\eta_k \cap G_T$, $1 \leq k \leq M$, or located on $V_j \cap G$, $j \in E$, falls inside at least one of the rectangles $\Pi_{k(p)}$, $1 \leq k(p) \leq M$, depending on P, and the distance from P to $G_T \bigcap \eta_{k(p)}$ is not less than some constant $\kappa_0 > 0$ independent of P.

The quantity κ_0 is called a *depth* of gluing of the rectangles Π_k, $k = 1(1)M$ (see Volkov [11]). We introduce the parameter $h \in (0, \kappa_0/2]$ and define a square grid on Π_k, $k = 1(1)M$, with maximal possible step $h_k \leq \min\{h, \min\{a_{1k}, a_{2k}\}/2\}$ such that the boundary η_k lies entirely on the grid lines. Let Π_k^h be the set of grid nodes on Π_k, let η_k^h be the set of nodes on η_k, and let $\overline{\Pi}_k^h = \Pi_k^h \cup \eta_k^h$. We denote the set of nodes on the closure of $\eta_k \cap G_T$ by η_{k0}^h, the set of nodes on t_j by t_j^h, and the set of remaining nodes on η_k by η_{k1}^h We also introduce a natural number n and the quantities $n(j) = \max\{4, [\alpha_j n]\}$, $\beta_j = \alpha_j \pi / n(j)$, and $\theta_j^m = (m - 1/2)\beta_j$, $j \in E$, $1 \leq m \leq n(j)$. On the arc V_j we choose the points (r_{j2}, θ_j^m), $1 \leq m \leq n(j)$, and denote the set of these points by V_j^n.

From the estimation (2.29) in Volkov [8] follows the existence of the positive constants n_0 and σ such that, for $n \geq n_0$,

$$\max_{(r_j,\theta_j)\in\overline{T}_j^3} \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q) \leq \sigma < 1. \tag{6}$$

Let

$$\omega^{h,n} = \left(\cup_{k=1}^{M}\eta_{k0}^{h}\right) \cup (\cup_{j\in E}V_j^n), \quad \overline{G}_T^{h,n} = \omega^{h,n} \cup (\cup_{k=1}^{M}\overline{\Pi}_k^h).$$

We define the matching operator S^2 at each point $P \in \omega^{h,n}$ in the following way. We consider the set of all rectangles $\{\Pi_k\}$ in the intersections of which the point P lies, and we choose one of these rectangles $\Pi_{k(P)}$ part of whose boundary, situated in G^T is furthest away from P. The value S^2u at the point P is computed according to the values of the function at the four vertices P_k, $k = 1,2,3,4$, of the closure of the cell, containing the point P, of the grid constructed on $\overline{\Pi}_{k(P)}$, by multilinear interpolation in the directions of the grid lines. Thus, S^2u has the expression

$$S^2u \equiv \sum_{\mu=1}^{4} \lambda_\mu u_\mu, \tag{7}$$

where $u = u(P)$, $u_\mu = u(P_\mu)$, and

$$\lambda_\mu \geq 0, \quad \sum_{\mu=1}^{4} \lambda_\mu = 1. \tag{8}$$

We define on $\Pi_k^h,\ 1 \leq k \leq M$ the operator A of calculating the arithmetic mean of the function at the four neighboring points of the same net.

Consider the system of linear algebraic equation

$$u_h = Au_h \quad \text{on} \quad \Pi_k^h, \tag{9}$$

$$u_h = \varphi_m \quad \text{on} \quad \eta_{k1}^h \cap \gamma_m, \tag{10}$$

$$u_h(r_j,\theta_j) = Q_j(r_j,\theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u_h(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q) \quad \text{on} \quad t_j^h, \tag{11}$$

$$u_h = S^2u_h \quad \text{on} \quad \omega^h, \tag{12}$$

where $1 \leq k \leq M$, $j \in E$.

Definition 3. The solution of the system (9)-(12) is called a numerical solution of the problem (1) on $\overline{G}_T^{h,n}$.

Definition 4. We consider the sector $T_j^* = T_j(r_j^*)$, where $r_j^* = (r_{j2} + r_{j3})/2$, $j \in E$. Let u_h be the solution of the system (9)-(12). The function

$$U_h(r_j, \theta_j) = Q_j(r_j, \theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q)(u_h(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q)), \quad (13)$$

defined on T_j^*, is called an approximate solution of the problem (1) on the closed block $\overline{T_j^3}$, $j \in E$.

Definition 5. The system (9)-(13) is called the system of block-grid equations.

4. Analysis of the System of Block-Grid Equations

4.1. Error for the Finite Difference Solution on Rectangular Domain

Let $G \equiv \Pi = \{(x, y) : 0 < x < a, 0 < y < b\}$ be a rectangle, a/b be rational, γ_j $j = 1, 2, 3, 4$ be the sides, including the ends, enumerated counterclockwise starting from the left side ($\gamma_0 \equiv \gamma_4$, $\gamma_5 \equiv \gamma_1$), and let $\gamma = \cup_{j=1}^4 \gamma_j$, be the boundary of Π. We consider the boundary value problem

$$\Delta u = 0 \text{ on } G, \quad u = \varphi_j(s) \text{ on } \gamma_j, \ j = 1, 2, 3, 4, \quad (14)$$

where φ_j is given function of the arc length s taken along γ and $\varphi_j \in C_{1,1}(\gamma_j)$. Furthermore, at the vertices $A_j = \gamma_{j-1} \cap \gamma_j$, $(s = s_j)$ the continuity condition $\varphi_{j-1}(s_j) = \varphi_j(s_j)$ is satisfied.

Let $h > 0$, with $a/h \geq 2$, $b/h \geq 2$ be integers. We assign Π^h, a square net on Π, with step h, obtained with the lines x, $y = 0, h, 2h, \ldots$. Let γ_j^h be a set of nodes on the interior of γ_j, and let

$$\dot{\gamma}_j^h = \gamma_j \cap \gamma_{j+1}, \quad \gamma^h = \cup_{j=1}^4 (\gamma_j^h \cup \dot{\gamma}_j^h), \quad \overline{\Pi}^h = \Pi^h \cup \gamma^h.$$

Let u_h be the solution of the finite difference problem:

$$u_h = Au_h \text{ on } \Pi^h, \quad u_h = \varphi_j \text{ on } \gamma_j^h.$$

The next theorem is proved by Volkov [13].

Theorem 6. *If* $\varphi_j \in C_{1,1}(\gamma_j)$ *and* $\varphi_{j-1}(s_j) = \varphi_j(s_j)$, $j = 1, 2, 3, 4$, *then*

$$\max_{\overline{\Pi}^h} |u_h - u| \leq ch^2(|\ln h| + 1),$$

where u *is the exact solution of the problem (14), and* c *is a constant independent of* h.

4.2. Analysis of the Block-Grid Equations

Theorem 7. *There is a natural number* n_0 *such that, for all* $n \geq n_0$ *the system (9)-(12) has a unique solution.*

Proof. The proof is obtained on the basis of (6)-(8), by analogy with Dosiyev [3]. □

Let

$$\varepsilon_h = u_h - u, \tag{15}$$

where u_h is a solution of system (9)-(12), and u is the trace on $\overline{G}_T^{h,n}$ of the solution of (1). On the basis of (1), (9)-(12) and (15) the error ε_h satisfies the system of difference equations

$$\begin{aligned}
\varepsilon_h &= A\varepsilon_h + r_h^1 \text{ on } \Pi_k^h, \\
\varepsilon_h &= 0 \text{ on } \eta_{k1}^h, \\
\varepsilon_h(r_j, \theta_j) &= \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q)\varepsilon_h(r_{j2}, \theta_j^q) + r_{jh}^2, (r_j, \theta_j) \in t_j^h, \qquad (16) \\
\varepsilon_h &= S^2\varepsilon_h + r_h^3 \text{ on } \omega^{h,n},
\end{aligned}$$

where $1 \leq k \leq M$, $j \in E$,

$$\begin{aligned}
r_h^1 &= Au - u \text{ on } \cup_{k=1}^M \Pi_k^h, \qquad (17) \\
r_{jh}^2 &= \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q)(u(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q)) \\
&\quad -(u(r_j, \theta_j) - Q_j(r_j, \theta_j)) \text{ on } \cup_{j\in E} t_j^h, \qquad (18) \\
r_h^3 &= S^2u - u \text{ on } \omega^{h,n}. \qquad (19)
\end{aligned}$$

In what follows and for simplicity, we will denote constants which are independent of h by c.

Lemma 8. *There exists a natural number n_0 such that, for all $n = \max\{n_0, \left[\ln^{1+\kappa}(h^2 |\ln h|)^{-1}\right] + 1\}$, where $\kappa > 0$ is a fixed number,*

$$\max_{j\in E} \left|r_{jh}^2\right| \le ch^2(|\ln h| + 1). \tag{20}$$

Proof. On the basis of (18), Lemma 2 and by virtue of $r_{j3} = (r_{j2}+r_{j4})/2 < r_{j2}$, we have

$$\begin{aligned}\left|r_{jh}^3\right| \le & \left|\beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q)(u(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q))\right. \\ & \left. - \int_0^{\alpha_j \pi} R_j(r_j, \theta_j, \eta)(u(r_{j2}, \eta) - Q_j(r_{j2}, \eta))d\eta\right| .\end{aligned}$$

From this and from the Lemma 2.10 of Volkov [8], we obtain

$$\left|r_{jh}^2\right| \le c_j^0 \exp\left\{-d_j^0 n\right\}, \quad j \in E, \tag{21}$$

where c_j^0 and $d_j^0 > 0$ are constants, independent of n. Putting $c^0 = \max_{j\in E}\left\{c_j^0\right\}$, and $d = \min\left\{d_j^0\right\}$ from (21) we have

$$\max_{j\in E} \left|r_{jh}^2\right| \le c^0 \exp\left\{-d^0 n\right\}. \tag{22}$$

Let n_0 be a natural number to hold the inequality (6). Then, for

$$n = \max\left\{n_0, \ \left[\ln^{1+\kappa}(h^2 |\ln h|)^{-1}\right] + 1\right\},$$

where $\kappa > 0$ is a fixed number, we have the inequality (20). □

Since the set of points $\omega^{h,n}$ located from the vertices of the polygon G at the distance exceeding some positive quantity independent of h, then by virtue of (2), estimation (2.8) obtained in Volkov [12], from (19) we obtain

$$\max_{\omega^{h,n}} \left|r_h^3\right| \le ch^2(|\ln h| + 1). \tag{23}$$

Theorem 9. *There exists a natural number n_0 such that, for*

$$n = \max\left\{n_0, \ \left[\ln^{1+\kappa}(h^2 |\ln h|)^{-1}\right] + 1\right\},$$

where $\kappa > 0$ is a fixed number,

$$\max_{\overline{G}_T^{h,n}} |u_h - u| \le ch^2(|\ln h| + 1).$$

Proof. Let us take an arbitrary rectangular grid $\Pi^h_{k^*}$ and let $t^h_{k^*j} = \overline{\Pi}^h_{k^*} \cap t^h_j$. Let $t^h_{k^*j} \neq \emptyset$, and v_h be a solution of system (16) in the case when the discrepancies r^1_h, r^2_h, r^3_{jh}, and r^4_h in $\overline{\Pi}^h_{k^*}$ are the same as in (17)-(19), but are zero in $\overline{G}^{h,n}_T \setminus \overline{\Pi}^h_{k^*}$. It is obvious that

$$W = \max_{\overline{G}^{h,n}_T} |v_h| = \max_{\overline{\Pi}^h_{k^*}} |v_h| . \tag{24}$$

We represent the function v_h on $\overline{G}^{h,n}_T$ in the form

$$v_h = \sum_{\kappa=1}^{4} v^\kappa_h, \tag{25}$$

where the functions v^κ_h, $\kappa = 2, 3, 4$ are defined on $\overline{\Pi}^h_{k^*}$ as a solution of the system of equations

$$\begin{aligned} v^2_h &= Av^2_h \text{ on } \Pi^h_{k^*}, v^2_h = 0 \text{ on } \eta^h_{k^*1}, \\ v^2_h(r_j, \theta_j) &= r^2_{jh}, (r_j, \theta_j) \in t^h_{k^*j}, \quad v^2_h = 0 \text{ on } \omega^{h,n}; \end{aligned} \tag{26}$$

$$\begin{aligned} v^3_h &= Av^3_h \text{ on } \Pi^h_{k^*}, v^3_h = 0 \text{ on } \eta^h_{k^*1}, \\ v^3_h(r_j, \theta_j) &= 0, (r_j, \theta_j) \in t^h_{k^*j}, \quad v^3_h = r^3_h \text{ on } \omega^{h,n}; \end{aligned} \tag{27}$$

$$\begin{aligned} v^4_h &= Av^4_h + r^1_h \text{ on } \Pi^h_{k^*}, v^4_h = 0 \text{ on } \eta^h_{k^*1}, \\ v^4_h(r_j, \theta_j) &= 0, (r_j, \theta_j) \in t^h_{k^*j}, \quad v^4_h = 0 \text{ on } \omega^{h,n}, \end{aligned} \tag{28}$$

with

$$v^\kappa_h = 0, \kappa = 2, 3, 4 \text{ on } \overline{G}^{h,n}_T \setminus \overline{\Pi}^h_{k^*}. \tag{29}$$

Hence according to (25)-(29) the function v^1_h satisfies the system of equations

$$\begin{aligned} v^1_h &= Av^1_h \text{ on } \Pi^h_k, \quad v^1_h = 0 \text{ on } \eta^h_{k1}, \\ v^1_h(r_j, \theta_j) &= \beta_j \sum_{q=1}^{n(j)} R^{(q)}_j(r_j, \theta_j) \sum_{\kappa=1}^{4} v^\kappa_h(r_{j2}, \theta^q_j), \quad (r_j, \theta_j) \in t^h_j, \\ v^1_h &= S^2 \left(\sum_{\kappa=1}^{4} v^\kappa_h \right) \text{ on } \eta^h_{k0}, \quad 1 \leq k \leq M, \ j \in E, \end{aligned} \tag{30}$$

where the functions $v^\kappa_h, \kappa = 2, 3, 4$ are assumed to be known.

Taking into account (20) and (23), on the basis of (26), (27), (29), and the principle of maximum, we have

$$W_2 = \max_{\overline{G}^{h,n}_T} \left| v^2_h \right| \leq ch^2(|\ln h| + 1), \tag{31}$$

$$W_3 \;=\; \max_{\overline{G}_T^{h,n}} \left| v_h^3 \right| \leq ch^2(|\ln h| + 1). \tag{32}$$

The function v_h^4 being a solution of the system (28) with (29) is the error of finite difference solution, with step $h_{k^*} \leq h$, of the Dirichlet problem for Laplace's equation on Π_{k^*}. Then, by virtue of (29) and Theorem 6, we obtain

$$W_4 = \max_{\overline{G}_T^{h,n}} \left| v_h^4 \right| = \max_{\overline{\Pi}_{k^*}^h} \left| v_h^4 \right| \leq ch^2(|\ln h| + 1). \tag{33}$$

We estimate the function v_h^1, which is, according to Theorem 7, the unique solution of system (30). On the basis of (6)-(8) and the gluing condition of the rectangles Π_k, $k = 1, 2, ..., M$, from (30) by means of Volkov [11], there exists a real number λ^*, $0 < \lambda^* < 1$, independent of h, such that for $n = \max\left\{n_0, \left[\ln^{1+\kappa}(h^2 |\ln h|)^{-1}\right] + 1\right\}$, we have

$$W_1 = \max_{\overline{G}_T^{h,n}} \left| v_h^1 \right| \leq \lambda^* W + \sum_{i=2}^{4} \max_{\overline{G}_T^{h,n}} \left| v_h^i \right|. \tag{34}$$

From (24), (25), (31)-(34), we obtain

$$W = \lambda^* W + 2\sum_{i=2}^{4} W_i \leq \lambda^* W + ch^2(|\ln h| + 1), \quad 0 < \lambda^* < 1,$$

i.e.,

$$W = \max_{\overline{G}_T^{h,n}} |v_h| \leq ch^2(|\ln h| + 1). \tag{35}$$

In the case, when $t_{k^*j}^h \equiv \emptyset$ the function $v_h^2 \equiv 0$ on $\overline{G}_T^{h,n}$ and the inequality (35) holds true.

Since the number of grid rectangles in $\overline{G}_T^{h,n}$ is finite, for the solution of (16) we have

$$\max_{\overline{G}_T^{h,n}} |\varepsilon_h| \leq ch^2(|\ln h| + 1). \qquad \square$$

4.3. Convergence of the Approximate Solution on Blocks

We consider the question of convergence of function $U_h(r_j, \theta_j)$ defined by the formula (13). Taking into account the properties of functions $Q_j(r_j, \theta_j), j \in E$ and the fact that $R_j(r_j, 0, \eta) = R_j(r_j, \alpha_j\pi, \eta) = 0$, the function $U_h(r_j, \theta_j)$ is bounded, harmonic on T_j^* and continuous up to its boundary, except for the vertex A_j when the specified boundary values are discontinuous at A_j. In addition, on the rectilinear parts of the boundary of T_j^*, except, maybe, the vertex A_j, function $U_h(r_j,\theta_j)$ satisfies the boundary conditions defined in (1).

Theorem 10. *There is a natural number n_0, such that for $n = \max\{n_0, \left[\ln^{1+\kappa}(h^2\,|\ln h|)^{-1}\right] + 1\}, \kappa > 0$ is a fixed number, the following inequalities are valid:*

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}(U_h(r_j,\theta_j) - u(r_j,\theta_j))\right| \le c_p h^2(|\ln h| + 1) \quad \text{on} \;\; \overline{T_j^3}, \tag{36}$$

first, for integer $1/\alpha_j$ when $p \ge 1/\alpha_j$, second, for any $1/\alpha_j$ when $p = 0$;

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}(U_h(r_j,\theta_j) - u(r_j,\theta_j))\right| \le c_p h^2(|\ln h| + 1)/r^{p-1/\alpha_j} \quad \text{on} \;\overline{T_j^3}, \tag{37}$$

for any $1/\alpha_j$, if $0 \le p < 1/\alpha_j$;

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}(U_h(r_j,\theta_j) - u(r_j,\theta_j))\right| \le c_p h^2(|\ln h| + 1)/r^{p-1/\alpha_j} \quad \text{on} \;\; \overline{T_j^3}\backslash A_j, \tag{38}$$

for noninteger $1/\alpha_j$, when $p > 1/\alpha_j$. Everywhere $0 \le q \le p$, u is a solution of the problem (1).

Proof. On the bases of (13) and Lemma 2, on the closed block $\overline{T_j^*}, j \in E$ we have

$$\begin{aligned} U_h(r_j,\theta_j) - u(r_j,\theta_j) = {}& \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)) \\ & - \int_0^{\alpha_j\pi} R_j(r_j,\theta_j,\eta)(u(r_{j2},\eta) - Q_j(r_{j2},\eta))d\eta \\ & + \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u_h(r_{j2},\theta_j^q) - u(r_{j2},\theta_j^q))\,. \end{aligned} \tag{39}$$

Since $r_j^* = (r_{j2} + r_{j3})/r_{j2}$, then for $n = \left[\ln^{1+\kappa}(h^2\,|\ln h|)^{-1}\right] + 1$, $\kappa > 0$ is a fixed number, we obtain

$$\left|\beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u(r_{j2},\theta_j^q) - Q_j(r_j,\theta_j^q))\right.$$

$$- \int_0^{\alpha_j \pi} R_j(r_j, \theta_j, \eta)(u(r_{j2}, \eta) - Q_j(r_{j2}, \eta))d\eta \Bigg| \\ \leq \quad ch^2(|\ln h| + 1), \quad \text{on} \quad \overline{T}_j^*, j \in E. \tag{40}$$

On the basis of (6) and Theorem 9 for $n = \max\left\{n_0, \left[\ln^{1+\kappa}(h^2 |\ln h|)^{-1}\right] + 1\right\}$ we have

$$\left|\beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q)(u_h(r_{j2}, \theta_j^q) - u(r_{j2}, \theta_j^q))\right| \leq ch^2(|\ln h| + 1), \\ \text{on } \overline{T_j^*}, j \in E. \tag{41}$$

From (39)-(41) for all $n = \max\left\{n_0, \ \left[\ln^{1+\kappa}(h^2 |\ln h|)^{-1}\right] + 1\right\}$ we obtain

$$|U_h(r_j, \theta_j) - u(r_j, \theta_j)| \leq ch^2(|\ln h| + 1), \quad \text{on } \overline{T_j^*}, j \in E. \tag{42}$$

Let

$$\varepsilon_h(r_j, \theta_j) = U_h(r_j, \theta_j) - u(r_j, \theta_j) \quad \text{on } \overline{T_j^*}\ , j \in E. \tag{43}$$

From (13), (43), and Remark 1 follows that the function $\varepsilon_h(r_j, \theta_j)$ is continuous on $\overline{T_j^*}$, and is a solution of the boundary value problem

$$\begin{aligned} \Delta\varepsilon &= 0 \text{ on } T_j^*, \\ \nu_m \varepsilon_h + \overline{\nu}_m (\varepsilon_h)'_n &= 0 \text{ on } \gamma_m \cap \overline{T_j^*}, m = j-1, j, \\ \varepsilon_h(r_j^*, \theta_j) &= U_h(r_j^*, \theta_j) - u(r_j^*, \theta_j), \quad 0 \leq \theta_j \leq \alpha_j \pi. \end{aligned} \tag{44}$$

Since $T_j^3 \subset \overline{T_j^*}, j \in E$, taking into account (42)-(44), from the Lemma 6.12 in Volkov [9] follows all inequalities of Theorem 10. ☐

5. The Use of Schwarz's Alternating Method to Solve the System of Block-Grid Equations

According to Definition 3 and Definition 4, the approximate solution of problem (1) must first be found in the domain $\overline{G}_T^{h,n}$ as the solution of the system of difference equations (9)-(12), and the solution itself and its derivatives of order p, $p = 1, 2, ...$, at any point of $\overline{T}_j^3$, $j \in E$, except may be the vertex A_j, can then be found using formula (13). Therefore, it is sufficient to justify the possibility of finding a solution of system (9)-(12) by Schwarz's alternating method.

We define the following classes $B_q, q = 1, 2, ..., q^*$, of rectangles $\Pi_k, k = 1, 2, ..., M$ (see Dosiyev [3]). Class B_1includes all rectangles whose intersection with the boundary γ of the polygon G contains a certain segment of positive length. Class B_2 contains all the rectangles which are not in the class B_1, whose intersection with rectangles of B_1 contains a segment of finite length, and so on.

Suppose we have a zero approximation $u_h^{(0)}$ to the exact solution u_h of (9)-(12). Finding $u_h^{(1)}$ for all $j \in E$ by the formula (11) on t_j^h and on η_{k0}, by (12), we solve the system (9), (10) on each grid $\overline{\Pi}_k^h$ of rectangles, first from class B_1, the from class B_2, and so on. The next iteration is similar.

Consequently, we have the sequence $u_h^{(1)}$, $u_h^{(2)}$, ..., defined as follows

$$\begin{aligned} u_h^{(m)}(r_j, \theta_j) &= Q_j(r_j, \theta_j) \\ &\quad + \beta_j \sum_{q=1}^{n(j)} R_j^{(q)}(r_j, \theta_j)(u^{(m-1)}(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q)) \quad \text{on} \quad t_j^h, \\ u_h^{(m)} &= S^2 u_h^{(m-1)} \quad \text{on} \quad \omega^{h,n}, \\ u_h^{(m)} &= A u_h^{(m)} \quad \text{on} \quad \Pi_k^h, \quad u_h^{(m)} = \varphi \quad \text{on} \quad \eta_{k1}^h, \end{aligned} \tag{45}$$

where, $1 \le k \le M, j \in E, m = 1, 2, ...$.

Theorem 11. *For $n = \max\left\{n_0, \left[\ln^{1+\kappa}(h^2 \left|\ln h\right|)^{-1}\right] + 1\right\}$ the system (9)-(12) can be solved by Schwarz's alternating method with any accuracy $\varepsilon > 0$ in a uniform metric with the number of iterations $O(\ln \varepsilon^{-1})$, independent of h and n, where n_0 and κ mean the same as in Theorem 10.*

Proof. The proof is obtained by analogy with the proof of Theorem 3 from Dosiyev [3]. □

Remark 12. If on the sides of right interior angles of polygon G the boundary functions are given also as an algebraic polynomials of s, then, without continuity condition $\varphi_{j-1}(s_j) = \varphi_j(s_j)$, the approximate solution in a neighborhood of vertices of these angles can be defined by the formula (13), and derivatives of any order can be found by its simple differentiation.

Remark 13. From the error estimation formula (37) of Theorem 10 follows that, the error of approximate solution on the block sectors decreases as $r_j^{1/\alpha_j} h^2(|\ln h| + 1)$, which gives an additional accuracy of the BGM near the singular points.

Remark 14. The method and results of this paper are valid for multiply connected polygons.

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