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DIMENSIONALITY REDUCING MULTIPLE INTEGRALS BY ALPHA-DENSE CURVES

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Abstract: We approximate the multiple integral of a non-negative real function f of class C^1 on the unit cube $[0,1]^n$, $n \geq 2$, by a simple one on the interval $[-1,1]$ by using a technique of densification of the region of quadrature. The curve that densifies is an α-dense curve called cosines curve. The integrand of the simple integral depends, as that of [8], on the Chebyshev polynomial of second kind. An estimation on the error generated by this reduction is also settled.

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1. Introduction

Given a real number $\alpha \geq 0$, an α-dense curve (see [6]) in a subset K of finite diameter of a metric space (E, d) is a continuous mapping $\gamma : [0,1] \to E$ whose image, now on noted γ^*, is contained in K and the distance $d(x, \gamma^*) \leq \alpha$ for any $x \in K$. In others words, an α-dense curve in K is a Peano Continuum

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(a connected, locally connected and compact set) γ^* such that the Hausdorff distance (see [5]) from K, $d_{\mathcal{H}}(\gamma^*, K) \leq \alpha$. For example, in the euclidean space R^{n+1}, $n \geq 1$, the cube $K_c = I^n \times [0, c]$, with $I = [0, 1]$, $c > 0$, is densified by the curves

$$\gamma_c^{(m)} = (\gamma_1^{(m)}, \gamma_2^{(m)}, ..\gamma_n^{(m)}, \gamma_{n+1}^{(m)}) : I \to K_c$$

defined by

$$\begin{aligned} \gamma_1^{(m)}(t) &= t\,, \\ \gamma_2^{(m)}(t) &= \frac{1}{2}(1 - \cos m\pi t)\,, \\ &\vdots \\ \gamma_n^{(m)}(t) &= \frac{1}{2}(1 - \cos m^{n-1}\pi t)\,, \\ \gamma_{n+1}^{(m)}(t) &= \frac{1}{2}(1 - \cos m^n \pi t)c\,, \end{aligned}$$

for each positive integer m.

It is easy to check that for each m the curve $\gamma_c^{(m)}$ has density $\frac{1}{m}\Phi(n)$ in K_c, where $\Phi(n)$ is a positive constant depending on n (for example, $\Phi(1) = 1$, $\Phi(2) = \sqrt{1 + \frac{\pi^2}{2^2}}$, etc.). Therefore their densities go to 0 as $m \to \infty$. In this case, K_c is said to be a densifiable set. Furthermore, as the length of $\gamma_c^{(m)}$ is given by

$$L(\gamma_c^{(m)}) = \int_0^1 \left\| (\gamma_c^{(m)})'(t) \right\| dt,$$

$(\gamma_c^{(m)})'(t)$ being the tangent vector, we have

$$\lim_{m\to\infty} L(\gamma_c^{(m)}) \frac{1}{m^n} = c,$$

which implies that the volume of K_c, noted vol(K_c), may be computed by the above limit. That is, the formula

$$\mathrm{vol}\,(K_c) = \lim_{m\to\infty} L(\gamma_c^{(m)}) \frac{1}{m^n} \tag{1}$$

follows (for more details see [7] and [8]).

Now, our purpose is to use the above formula for reducing the multiple integral of a non-negative real function f of class C^1 on the unit cube $[0, 1]^n$, $n \geq 2$, to a simple one on the interval $[-1, 1]$, as we shall see below. Furthermore, the error of the approximation will be estimated as function depending on the integrand f, on the α-dense curve that densifies I^n (by means of its densification parameter m) and, of course, on the dimension n.

We shall see that this reductional technique is total in the sense that the final dimension es 1, so it may be compared with that of [4].

2. The Theorem of Approximation for the Cosines Curve

Since the above curves will be frecuently used, we start with the following definition.

Definition 1. For each integer $m \geq 1$, the curve $\Gamma_{I^n}^{(m)} : [0,1] \to I^n$, $n \geq 2$, defined by $\Gamma_{I^n}^{(m)}(t) = (\gamma_1^{(m)}(t), \gamma_2^{(m)}(t), ..., \gamma_n^{(m)}(t))$ with $\gamma_1^{(m)}(t) = t$ and $\gamma_k^{(m)}(t) = \frac{1}{2}(1 - \cos m^{k-1}\pi t)$, $2 \leq k \leq n$, will be called the cosines curve of the order m in the unit cube $I^n = [0,1]^n$ of $R^n, n \geq 2$.

Clearly, the cosines curve $\Gamma_{I^n}^{(m)}(t)$ is a particular case of the preceding curves $\gamma_c^{(m)}$ (it is enough to take $c = 1$ and to change n by $n-1$). Therefore, the density of $\Gamma_{I^n}^{(m)}(t)$ in I^n, for $n \geq 2$, is $\frac{1}{m}\Phi(n-1)$. We can also extend that for the trivial case $n = 1$ by defining $\Gamma_I^{(m)}(t) \equiv t$. Thus the density of this curve in I, formally written $\frac{1}{m}\Phi(0)$, is obviously 0, and consequently $\Phi(0) = 0$.

In [8] (Theorem 10, p. 63) we gave a reduction integral formula by using a polynomial curve, namely, the Chebyshev curve. Now the cosines curve, yields a similar result with a common peculiarity: the appearance of the Chebyshev polynomial of the second kind in the integrand.

Theorem 2. *Let f be a real non-negative function of class C^1 on I^n and $\Gamma_{I^n}^{(m)}$ the cosines curve of the order m in $I^n, n \geq 1$. Thus the integral*

$$\int_{I^n} f(x_1, ..., x_n) dx_1 ... dx_n \tag{2}$$

may be approximated, for large enough m, by

$$\frac{1}{2} \int_{-1}^{1} \left| U_{m^n - 1}(s) \right| f^*(\frac{\arccos s}{\pi}) ds, \tag{3}$$

where $U_{m^n-1}(s)$ is the Chebyshev polynomial of the second kind of degree $m^n - 1$ and $f^* = f \circ \Gamma_{I^n}^{(m)}$.

Proof. Since the cosines curve densifies any cube $I^n \times [0,c]$ with arbitrary small density, by the continuity of f, it is clear that the curve

$$\gamma_f^{(m)}(t) = (t, \frac{1}{2}(1-\cos m\pi t), ..., \frac{1}{2}(1-\cos m^{n-1}\pi t), \\ \frac{1}{2}(1-\cos m^n \pi t) f^*(t)), \quad (4)$$

briefly noted $(\Gamma_{I^n}^{(m)}(t), \frac{1}{2}(1-\cos m^n\pi t)f^*(t))$, densifies the domain of quadrature

$$\mathcal{D}_f = \left\{(x,y) \in R^{n+1} : x \in I^n, 0 \le y \le f(x)\right\}$$

with arbitrary small density.

By approximating f by step functions, their integrals are determined by applying the formula that determines the volume of a cube, which has been exhibed in the Introduction. Hence for m enough large

$$\int_{I^n} f(x_1,...,x_n)dx_1...dx_n \approx L(\gamma_f^{(m)})\frac{1}{m^n} = \frac{1}{m^n}\int_0^1 \left\|(\gamma_f^{(m)})'(t)\right\| dt$$

$$= \frac{1}{m^n}\int_0^1 \left[\begin{array}{c} 1+(\frac{1}{2}m\pi \sin m\pi t)^2 + ... + (\frac{1}{2}m^{n-1}\pi \sin m^{n-1}\pi t)^2 + \\ ((\frac{1}{2}m^n\pi \sin m^n\pi t)f^*(t) + \frac{1}{2}(1-\cos m^n\pi t)f^{*\prime}(t))^2 \end{array}\right]^{\frac{1}{2}} dt.$$

Noticing n is fixed , the above integral can be substituted by

$$\frac{\pi}{2}\int_0^1 |\sin m^n\pi t| \, f^*(t)dt. \quad (5)$$

Now, by the change of variable $\pi t = \arccos s$ and according that the Chebyshev polynomial of second kind $U_j(s) = \frac{(T_{j+1}(s))'}{j+1}$, with $T_k(s) = \cos(k \arccos s)$, $k \geq 0$, $s \in [-1,1]$, the Chebyshev polynomial of first kind, the integral (4) becomes

$$\frac{1}{2}\int_{-1}^1 \left|U_{m^{n-1}}(s)\right| f^*(\frac{\arccos s}{\pi})ds\,, \quad (6)$$

and so the proof is complete. □

3. The Precision of the Reduction

From the above theorem we have obtained the approximation formula

$$\int_{I^n} f(x_1,...,x_n)dx_1...dx_n \approx \frac{\pi}{2}\int_0^1 |\sin m^n\pi t| \, f^*(t)dt, \quad (7)$$

which allows us to reduce directly the multiple integral to a simple one, so the dimensional reduction is total. However it is necessary to estimate the error in the approximation, noted $r^{(n)}_{m,f}$, and defined by

$$r^{(n)}_{m,f} = \int_{I^n} f(x_1,...,x_n)dx_1...dx_n \ - \ \frac{1}{2}\int_{-1}^{1} \left|U_{m^{n-1}}(s)\right| f^*(\frac{\arccos s}{\pi})ds\,,$$

or equivalently

$$r^{(n)}_{m,f} = \int_{I^n} f(x_1,...,x_n)dx_1...dx_n \ - \ \frac{\pi}{2}\int_{0}^{1} \ \left|\sin m^n \pi t\right| f^*(t)dt. \qquad (8)$$

Now we need the concept of algebraic precision.

Definition 3. Fixed the dimension n, we say that the formula of integral reduction (1) has algebraic precision of order $k \geq 0$ if and only if the error $r^{(n)}_{m,f} = 0$, for all polynomial f of degree less or equal to k and for all positive integer m.

The first result on the algebraic precision of the formula of reduction (1) is established.

Proposition 4. *For any dimension n the formula of integral reduction has algebraic precision of order 0. Furthermore, in dimension 1 the formula has algebraic precision of order 1.*

Proof. Noticing for all positive integers m, n the integral

$$\frac{\pi}{2}\int_{0}^{1} \ \left|\sin m^n \pi t\right| dt = 1,$$

the error vanishes for any f constant. Hence the first part is proved. For the second part, assume without loss of generality that $f(t) = t$. Thus

$$\int t \sin m\pi t dt = \frac{\sin m\pi t}{m^2\pi^2} - \frac{t\cos m\pi t}{m\pi}$$

and since $\frac{\sin m\pi t}{m^2\pi^2}$ vanishes for any value of t belonging to the set $\left\{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m}{m}\right\}$, the integral

$$\frac{\pi}{2}\int_{0}^{1} \ \left|\sin m\pi t\right| f^*(t)dt = \frac{\pi}{2} \ \int_{0}^{1} t \ \left|\sin m\pi t\right| dt$$
$$= \frac{\pi}{2}\sum_{i=1}^{m}(-1)^{i-1}\int_{\frac{i-1}{m}}^{\frac{i}{m}} t \ \sin m\pi t dt = \frac{\pi}{2}\sum_{i=1}^{m}(-1)^{i-1}\left[-\frac{t\cos m\pi t}{m\pi}\right]_{\frac{i-1}{m}}^{\frac{i}{m}}$$

$$= \frac{1}{2m^2}\sum_{i=1}^{m}(2i-1) = \frac{1}{2}. \quad (9)$$

As trivially $\int_I f(t)dt = \frac{1}{2}$, from (1) the error $r_{m,f}^{(1)} = 0$, for any m. This proves that the formula has algebraic precision of order, at least 1.

Taking now $f(t) = t^2$, a primitive

$$\int t^2 \sin(m\pi t)dt = 2t\frac{\sin m\pi t}{m^2\pi^2} - \frac{t^2\cos m\pi t}{m\pi} + \frac{2\cos m\pi t}{m^3\pi^3},$$

so

$$\frac{\pi}{2}\int_0^1 |\sin m\pi t|\, f^*(t)dt = \frac{\pi}{2}\sum_{i=1}^{m}(-1)^{i-1}\left[\frac{2\cos m\pi t}{m^3\pi^3} - \frac{t^2\cos m\pi t}{m\pi}\right]_{\frac{i-1}{m}}^{\frac{i}{m}}$$

$$= \frac{1}{2m}\sum_{i=1}^{m}\left(\frac{i^2+(i-1)^2}{m^2} - \frac{4}{m^2\pi^2}\right) = \frac{1}{3} + \frac{\pi^2-12}{6m^2\pi^2}. \quad (10)$$

On the other hand, as $\int_I f(x)dx = \frac{1}{3}$, (2) implies that the error $r_{m,f}^{(1)} = \frac{12-\pi^2}{6m^2\pi^2} = O(\frac{1}{m^2})$. Consequently, the formula has algebraic precision of order 1, as claimed. □

The general estimation of the error is given in the following result.

Theorem 5. *For any dimension n, the error in the integral reduction formula is an $O(\frac{1}{m})$.*

Proof. Let m be a positive integer, express the simple integral as the sum

$$\frac{\pi}{2}\int_0^1 |\sin m^n\pi t|\, f^*(t)dt = \frac{\pi}{2}\sum_{i=1}^{m^n}\int_{\frac{i-1}{m^n}}^{\frac{i}{m^n}}(-1)^{i-1}(\sin m^n\pi t)f^*(t)dt\ . \quad (11)$$

Now, since for each $1 \le i \le m^n$ the function $(-1)^{i-1}(\sin m^n\pi t)$ is non-negative on the interval $\left[\frac{i-1}{m^n}, \frac{i}{m^n}\right]$, by applying the first mean value for integrals, (1) is equal to

$$\frac{\pi}{2}\sum_{i=1}^{m^n} f^*(\xi_i)\int_{\frac{i-1}{m^n}}^{\frac{i}{m^n}}(-1)^{i-1}(\sin m^n\pi t)dt, \quad (12)$$

with $\xi_i \in \left[\frac{i-1}{m^n}, \frac{i}{m^n}\right]$.

Noticing that, for all $1 \leq i \leq m^n$, the integral $\int_{\frac{i-1}{m^n}}^{\frac{i}{m^n}} (-1)^{i-1}(\sin m^n \pi t)dt = \frac{2}{m^n \pi}$, by substituting that in (2) we are led to

$$\frac{\pi}{2}\int_0^1 |\sin m^n \pi t|\, f^*(t)dt = \frac{1}{m^n}\sum_{i=1}^{m^n} f^*(\xi_i). \tag{13}$$

On the other hand, from the additivity of the integral,

$$\int_{I^n} f(x_1, ..., x_n)dx_1...dx_n = \sum_{i=1}^{m^n} \mathrm{Vol}\,(I_i^n) f(X_i), \tag{14}$$

for some points X_i belonging to I_i^n, where I_i^n denotes the i-th-subcube, $1 \leq i \leq m^n$, of a partition of the unit cube I^n and $\mathrm{Vol}\,(I_i^n)$ is the measure of its length, area, volume , etc.

For $n = 1$ the cosines curve is $\Gamma_I^{(m)}(t) \equiv t$, with $t \in I$. The subintervals of the partition of I are taken as $I_i^1 = \left[\frac{i-1}{m}, \frac{i}{m}\right]$,$1 \leq i \leq m$, so $\mathrm{Vol}\,(I_i^1) = \frac{1}{m}$ and the points $X_i, \xi_i \in I_i^1$. Thus, from (3), (4) and using the Mean Value Theorem, the error is

$$\left| r_{m,f}^{(n)} \right| \leq \frac{1}{m} \|f\|_1 \sum_{i=1}^{m} |X_i - \xi_i| \leq \frac{1}{m} \|f\|_1 \,,$$

where $\|f\|_1 \equiv \sup\{|f(x)| : x \in I\} + \sup\{|f'(x)| : x \in I\}$. Therefore, in this case, the theorem follows.

Now, one assumes $n \geq 2$ and m even. Take a partition of I^n formed by subcubes I_i^n obtained by dividing each edge of I^n, except the last, in m equal subintervals $\left[0, \frac{1}{m}\right], \left[\frac{1}{m}, \frac{2}{m}\right], ..., \left[\frac{m-1}{m}, \frac{m}{m}\right]$. The last edge is divided in the subintervals $\left[\frac{1}{2}(1 - \cos\pi\frac{i-1}{m}), \frac{1}{2}(1 - \cos\pi\frac{i}{m})\right]$, $1 \leq i \leq m$. These subcubes I_i^n, $1 \leq i \leq m^n$, are arranged following the trajectory of the cosines curve $\Gamma_{I^n}^{(m)}$, considered as a physical motion, where t would be the time. Then, for each $1 \leq i \leq \frac{m}{2}$,

$$\mathrm{Vol}\,(I_i^n) = \frac{1}{m^{n-1}} a_i, \tag{15}$$

where

$$a_i = \frac{1}{2}(1 - \cos\pi\frac{i}{m}) - \frac{1}{2}(1 - \cos\pi\frac{i-1}{m}) = \sin\pi\frac{2i-1}{2m}\sin\frac{\pi}{2m}. \tag{16}$$

Observe that $\mathrm{Vol}\,(I_i^n) = \mathrm{Vol}\,(I_{m-(i-1)}^n)$, for $\frac{m}{2} + 1 \leq i \leq m$, and all is symmetrically repeated from m. Therefore, it is enough to examine the error

$$r_{m,f}^{(n)} = \frac{1}{m^{n-1}}\sum_{i=1}^{m^n}(a_i f(X_i) - \frac{1}{m} f^*(\xi_i)) = \frac{1}{m^{n-1}}\sum_{i=1}^{m^n}(a_i f(X_i) - \frac{1}{m} f(Y_i)), \tag{17}$$

for $1 \leq i \leq \frac{m}{2}$, where $Y_i \equiv \Gamma_{I^n}^{(m)}(\xi_i)$.

First we note that the point $Y_i = (y_{i,1}, y_{i,2}, ..., y_{i,n}) \in I_i^n$ for any $1 \leq i \leq \frac{m}{2}$. Indeed,

$$y_{i,1} = \xi_i \in \left[\frac{i-1}{m^n}, \frac{i}{m^n}\right] \subset \left[0, \frac{1}{m}\right],$$

and since the function $g(x) = x - \sin^2 \frac{\pi}{2}x \geq 0$ for $0 \leq x \leq \frac{1}{2}$,

$$y_{i,k} = \frac{1}{2}(1 - \cos m^{k-1}\pi\xi_i) \in \left[\frac{1}{2}(1 - \cos\pi\frac{i-1}{m^{n-k+1}}), \frac{1}{2}(1 - \cos\pi\frac{i}{m^{n-k+1}})\right]$$
$$\subset \left[0, \sin^2\frac{\pi}{2}\frac{i}{m^{n-k+1}}\right] \subset \left[0, \frac{i}{m^{n-k+1}}\right] \subset \left[0, \frac{1}{m}\right],$$

for all $2 \leq k \leq n-1$. Finally,

$$y_{i,n} = \frac{1}{2}(1 - \cos m^{n-1}\pi\xi_i) \in \left[\frac{1}{2}(1 - \cos\pi\frac{i-1}{m}), \frac{1}{2}(1 - \cos\pi\frac{i}{m})\right],$$

therefore, $Y_i \in I_i^n$, as claimed. Now, (7) is rewritten as

$$r_{m,f}^{(n)} = \frac{1}{m^{n-1}}\sum_{i=1}^{m^n}\left[a_i(f(X_i) - f(Y_i)) + (a_i - \frac{1}{m})f(Y_i)\right]. \qquad (18)$$

The Mean Value Theorem (see [3]) applied to f, turns (8) in

$$r_{m,f}^{(n)} = \frac{1}{m^{n-1}}\sum_{i=1}^{m^n}\left[a_i\nabla f(Z_i).(X_i - Y_i) + (a_i - \frac{1}{m})f(Y_i)\right], \qquad (19)$$

where $\nabla f(Z_i)$ denotes the gradient of f at a point Z_i of the linear segment determined by the points X_i, Y_i, and the dot (.) represents the usual inner product in R^n. Now let us write

$$r_{m,f}^{(n)} = s_{m,f}^{(n)} + t_{m,f}^{(n)} \quad , \qquad (20)$$

where

$$s_{m,f}^{(n)} = \frac{1}{m^{n-1}}\sum_{i=1}^{m^n} a_i\nabla f(Z_i).(X_i - Y_i) \ ; \ t_{m,f}^{(n)} = \frac{1}{m^{n-1}}\sum_{i=1}^{m^n}(a_i - \frac{1}{m})f(Y_i).$$

By the Cauchy inequality,

$$\left|s_{m,f}^{(n)}\right| \leq \frac{1}{m^{n-1}}\sum_{i=1}^{m^n} a_i \left\|\nabla f(Z_i)\right\| \left\|X_i - Y_i\right\|. \qquad (21)$$

Whenever the space $\mathcal{C}^1(I^n;R)$ is equipped with the norm

$$\|f\|_1 = \sup_{U\in I^n} |f(U)| + \sup_{V\in I^n} \|\nabla f(V)\|, \tag{22}$$

(21) involves

$$\left|s^{(n)}_{m,f}\right| \leq \frac{1}{m^{n-1}} \sum_{i=1}^{m^n} a_i \|f\|_1 \|X_i - Y_i\|. \tag{23}$$

Noticing the symmetry of the subcubes, (13) can be rewritten as

$$\left|s^{(n)}_{m,f}\right| \leq \frac{1}{m^{n-1}} 2m^{n-1} \sum_{i=1}^{\frac{m}{2}} a_i \|f\|_1 \|X_i - Y_i\| = 2\|f\|_1 \sum_{i=1}^{\frac{m}{2}} a_i \|X_i - Y_i\|. \tag{24}$$

Because of

$$\|X_i - Y_i\| \leq \left[(n-1)\frac{1}{m^2} + a_i^2\right]^{\frac{1}{2}} \leq \left[(n-1)\frac{1}{m^2} + \sin^2\frac{\pi}{2m}\right]^{\frac{1}{2}},$$

$$\text{for } 1 \leq i \leq \frac{m}{2} \text{ and } \sum_{i=1}^{\frac{m}{2}} a_i = \frac{1}{2},$$

from (24) it follows

$$\left|s^{(n)}_{m,f}\right| \leq \|f\|_1 \left[(n-1)\frac{1}{m^2} + \sin^2\frac{\pi}{2m}\right]^{\frac{1}{2}} = O(\frac{1}{m}). \tag{25}$$

On the other hand the second summand of (20)

$$t^{(n)}_{m,f} = \frac{1}{m^{n-1}} \sum_{i=1}^{m^n} (a_i - \frac{1}{m}) f(Y_i)$$
$$= \frac{1}{m^{n-1}} \left[\begin{array}{c} \sum_{i=1}^{\frac{m}{2}} (a_i - \frac{1}{m}) f(Y_i) + \\ + \sum_{i=\frac{m}{2}+1}^{m} (a_i - \frac{1}{m}) f(Y_i) \\ + \sum_{i=m+1}^{m+\frac{m}{2}} (a_i - \frac{1}{m}) f(Y_i) + ... \end{array} \right]. \tag{26}$$

Now, each summand of the above expression, e.g. $\sum_{i=1}^{\frac{m}{2}} (a_i - \frac{1}{m}) f(Y_i)$, is transformed by the summation parts Abel formula in

$$b_{\frac{m}{2}+1} f(Y_{\frac{m}{2}+1}) - b_1 f(Y_1) - \sum_{i=1}^{\frac{m}{2}} b_{i+1} \Delta f(Y_i), \tag{27}$$

by defining the sequence $b_i \equiv \frac{1}{2}(1-\cos\pi\frac{i-1}{m})-\frac{i}{m}$, and Δ denoting the difference operator, i.e., $\Delta f(Y_i) \equiv f(Y_{i+1})-f(Y_i)$. Thus, by (17) and repeating the preceding reasoning for the boundeness of $s_{m,f}^{(n)}$,

$$\sum_{i=1}^{\frac{m}{2}}(a_i-\frac{1}{m})f(Y_i)=\frac{1}{m}f(Y_1)-\sum_{i=1}^{\frac{m}{2}}b_{i+1}\left[f(Y_{i+1})-f(Y_i)\right]=O(\frac{1}{m}). \tag{28}$$

Therefore , from (16) $t_{m,f}^{(n)}=O(\frac{1}{m})$ and taking into account (10) and (15), the error

$$r_{m,f}^{(n)}=O(\frac{1}{m}),$$

which completes the proof of the theorem. □

4. Numerical Approach

We have implemented in programmation language C some routines for applying the above reductional technique consisting of substituting the multiple integral

$$\int_{I^n} f(x_1,...,x_n)dx_1...dx_n\ ,$$

by

$$\frac{\pi}{2}\int_0^1 \ |\sin m^n\pi t|\ f^*(t)dt$$

and to compute this one by the Simpson method. We have obtained the following numerical results.

Example 1. $\iiint_{I^3} xyzdxdydz = 0,125$, which approximation by the method is

$m=1$	$0,229166666666665$
$m=2$	$0,125000000000003$
$m=3$	$0,131228585931315$
$m=4$	$0,124999989472422$
$m=5$	$0,127106312029298$
$m=10$	$0,125000000676531$

Example 2. The probability integral

$$\frac{1}{\pi\sqrt{3}}\int_{I^2}\exp\left[-\frac{2}{3}\left(x^2-xy+y^2\right)\right]dxdy\approx 0,1377$$

is approximated by

$m=1$	$0,150819759377466$
$m=2$	$0,137928841413040$
$m=3$	$0,138507516885383$
$m=4$	$0,137731879675459$
$m=5$	$0,137978987126065$
$m=10$	$0,137697787718432$

Finally in dimension $n=5$.

Example 3. $\int_{I^5} x_1x_2x_3x_4x_5\ dx_1dx_2dx_3dx_4dx_5 = 0,03125$, approximated by

$m=1$	$0,150781250000001$
$m=2$	$0,031249999999999$
$m=3$	$0,032866488667339$
$m=4$	$0,031250002482228$
$m=5$	$0,031778737692331$
$m=14$	$0,031250000000003$

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