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INSTABILITY OF PERIODIC WAVES OF A VAN DER WAALS FLUID MODEL

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Abstract: We study a van der Waals fluid model and prove the existence of spatially-periodic travelling waves if and only if the velocity is zero. We prove the spectral instability of a stationary periodic wave: the spectral analysis is carried out by means of Floquet's theory. After introducing the Evans function $D(\lambda, \theta)$, the result about stability is achieved by means of a description of the zero set of D around the origin. This zero set is described at the leading order by a formula which involves a flux of a suitable first-order system of conservation laws derived by means of homogenisation procedure.

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1. Introduction

Consider the following relaxation model of fluid dynamics:

$$\begin{cases} v_t = u_x, \\ u_t + (v_{xx} + p)_x = 0, \\ p_t + a^2 u_x = \pi(v) - p; \end{cases} \tag{1.1}$$

where $u, v, p : \mathcal{R} \times (0, \infty) \to \mathcal{R}$ are the unknown functions; $x \in \mathcal{R}$, $t > 0$, a is a real constant and $\pi : \mathcal{R} \to \mathcal{R}$ is given.

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The system represents a Van der Waals gas model, in the sense that the equilibrium pressure π is a non-monotone function of the density. The third derivative of the specific volume v represents the capillarity term, which is necessary in the model, as the dissipation induced by the relaxation cannot account for all realistic phenomena. The function u is the velocity field and p represents the pressure (for a thorough analysis of van der Waals gas models we refer to Serre [4] and to Whitham [6]).

In the present paper we are interested in the spectral stability of space periodic travelling waves solutions to (1.1). More precisely, a travelling wave solution (TWS) with velocity $s \in \mathcal{R}$ is a real function of the form $F = F(x - st)$; after substituting in the equations of system (1.1), in order to prove the existence of periodic waves one has to study the existence of periodic solutions of the profile equations. We shall prove below that the system (1.1) admits non-trivial periodic TWS if and only if $s = 0$.

A general study of spectral stability of space-periodic, time-steady solutions to systems of conservation laws with diffusive terms, has been carried out by Serre in [5]: the spectral stability of a space periodic travelling wave has been investigated through Floquet's theory, by introducing the Evans function. In unifying the results proved previously by Serre in [4] and by Oh and Zumbrun in [2], the main theorem of [5] gives us a description of the zero set of the Evans function around the origin and establishes the link between this set and the flux of a first order system of conservation laws, derived by means of an homogenisation procedure. The hyperbolicity of the latter system is a necessary condition for spectral stability of periodic travelling waves.

By following the procedure adopted in [5], we shall investigate the spectral stability of non-trivial steady periodic waves of (1.1). In order to carry out the spectral analysis, we linearize the equations of (1.1) around a specific periodic wave. Thus, by means of the Floquet's theory for linear differential operators with periodic coefficients, we study the spectrum of the first order linearized operator associated with (1.1), by introducing the corresponding monodromy matrix and the Evans function. Moreover, we describe the behaviour of the Evans function in a neighbourhood of the origin. The specific form of the Evans function at leading order allows us to establish the result on the stability of periodic waves.

According to the results proved by Serre in [4] with different techniques for a Van der Waals gas model, we shall prove in Theorem 5.1 that if (u_0, v_0, p_0) is a non-trivial stationary periodic solution to (1.1), then (u_0, v_0, p_0) is spectrally unstable.

Moreover,we derive from the equations of (1.1) a first-order system of conservation laws describing the slow modulation of the periodic travelling waves. Our second main result, proved in Theorem 5.2, deals with the formula which describes the zero set of the Evans function at leading order in terms of a flux obtained from the equations in macroscopic variables.

The paper is organized as follows. In Section 2, we prove the existence of space-periodic time-steady solutions to (1.1). In the subsequent section we derive a system of three conservation laws, through homogenisation procedure in the equations of (1.1). Section 4 is devoted to the spectral analysis of the problem by means of Floquet's theory and to the analysis of the Evans function. In last section we prove the main theorems.

2. Existence of Periodic Waves

In the present section we shall prove the existence of space-periodic TWS to the system (1.1).

We begin by proving a necessary condition, which allows us to find a suitable surface where periodic travelling waves solutions do exist. The result is achieved by means of a suitable convex entropy-flux pair for the system (1.1), in such a way that a solution to (1.1) satisfies a dissipative inequality.

Lemma 2.1. *Let us consider the system (1.1). Assume that the function* $\pi : \mathcal{R} \to \mathcal{R}$ *is differentiable and that* $\pi'(y) > -a^2$*, for every* $y \in \mathcal{R}$*. If* (u, v, p) *is a periodic TWS to (1.1), then it holds true that* $p = \pi(v)$*.*

Proof. Let us fix a solution (u, v, p) to (1.1) and define the following functions:

$$\begin{aligned} \eta &= \tfrac{1}{2}(au+p)^2 + \tfrac{1}{2}(au-p)^2 + h(a^2v+p), \\ q &= \tfrac{1}{2}(au+p)^2 - \tfrac{1}{2}(au-p)^2; \end{aligned} \tag{2.1}$$

where h is a real function which is defined below. Let us differentiate the function η with respect to t and the function q with respect to x. By taking into account the equations of (1.1), we get

$$\eta_t + q_x = (\pi(v) - p)\left[2p + h'(a^2v+p)\right] - a^2(v_x^2)_t + 2a^2(-uv_{xx} + u_xv_x)_x. \tag{2.2}$$

Set

$$\begin{aligned} \tilde{\eta} &= \eta + a^2(v_x^2), \\ \tilde{q} &= q - 2a^2(-uv_{xx} + u_xv_x). \end{aligned} \tag{2.3}$$

Hence

$$\tilde{\eta}_t + \tilde{q}_x = (\pi(v) - p)[2p + h'(a^2 v + p)]. \tag{2.4}$$

Let us denote by $r(u, v, p)$ the r.h.s. in (2.4). In order to have a dissipativity relation, we look for a suitable function h in such a way that $r(u, v, p) \le 0$. Define the function $f : \mathcal{R} \to \mathcal{R}$ as $f(v) = a^2 v + \pi(v)$. Since $\pi'(y) > -a^2$, the function f is invertible.

Set $h(s) = -2 \int_0^s \pi(f^{-1}(z))dz$, for every $s \in \mathcal{R}$.

Thus $h(a^2 v + \pi(v)) = -2 \int_0^{a^2 v + \pi(v)} \pi(f^{-1}(z))dz$, and $h'(a^2 v + \pi(v)) + 2\pi(v) = 0$, for every $v \in \mathcal{R}$.

Moreover, the assumption $\pi'(y) > -a^2$ yields $h''(z) + 2 > 0$, for every $z \in \mathcal{R}$. By defining the function h as above, we obtain the following identity

$$h'(a^2 v + p) = h'(a^2 v + p) - h'(a^2 v + \pi(v)) - 2\pi(v) = h''(\psi)(p - \pi(v)) - 2\pi(v),$$

where $\psi = \psi(v, p)$ is a suitable real value. Hence,

$$r(u, v, p) = -(\pi(v) - p)^2 (h''(\psi) + 2) \quad \text{and} \quad r(v, u, p) \le 0.$$

Let $G = G(x - st)$ be a TWS to (1.1) where $G = (v, u, p)^t$. Therefore

$$(\tilde{\eta}(G))_t + (\tilde{q}(G))_x = (-s\tilde{\eta}'(G) + \tilde{q}'(G))G' = r(G). \tag{2.5}$$

Assume that the function G is a space periodic TWS to (1.1) with period X and consider (2.5). After integration, we get

$$\int_0^X (-s\tilde{\eta}'(G) + \tilde{q}'(G))G' dx = 0. \tag{2.6}$$

Therefore

$$\int_0^X r(G)dx = -\int_0^X (\pi(v) - p)^2 (h''(\psi) + 2)dx = 0, \tag{2.7}$$

and we derive the necessary condition of the lemma. □

Lemma 2.2. *Let us consider the system (1.1) under the assumptions of the previous Lemma. Assume moreover that the function π admits a first integral and denote it by Π. Define the function $F : \mathcal{R} \to \mathcal{R}$, as $F(z) = cz - 2\Pi(z) + C$, where c and C are real constants. If there are $\alpha, \beta \in \mathcal{R}$ such that the function F is positive on the interval (α, β), $F(\alpha) = F(\beta) = 0$ and $F'(\alpha) \ne 0, F'(\beta) \ne 0$, then there exists a unique non constant stationary periodic wave of (1.1) whose range equals the interval $[\alpha, \beta]$. Moreover, the system (1.1) admits non trivial periodic TWS if and only if $s = 0$.*

Proof. Because of the result of the previous lemma, we look for a periodic solution to (1.1) on the surface $p = \pi(v)$ of a 3-dimensional euclidean space. We obtain the following equations, where the prime denotes derivatives with respect to $x - st$:

$$\begin{cases} -sv' = u', \\ -su' + (v'' + \pi(v))' = 0, \\ s(\pi(v))' + a^2 sv' = 0. \end{cases} \tag{2.8}$$

In the case where $s \neq 0$, we derive only constant solutions since $\pi'(y) > -a^2$. If $s = 0$, we get

$$\begin{cases} u = \text{constant}, \\ (v'' + \pi(v))' = 0, \\ p = \pi(v). \end{cases} \tag{2.9}$$

Thus we have to find periodic solutions to the equation

$$(v'' + \pi(v))' = 0. \tag{2.10}$$

After integrating and multiplying by v' both sides of (2.10), we integrate again and obtain the following equation

$$(v')^2 = cv - 2\Pi(v) + C, \tag{2.11}$$

where c and C are suitable real constants. Because of our assumptions, the function on the r.h.s. of (2.11) is positive on the interval (α, β) and becomes equal zero in α and β. Hence the equation (2.10) admits a periodic solution $v : \mathcal{R} \to [\alpha, \beta]$, which assumes the extrema in α and β.

Moreover, v is the unique (up to translation of the argument) periodic solution to (2.10), which takes values in the interval $[\alpha, \beta]$. □

3. Asymptotic Analysis

In the present section an homogenisation procedure in the equations of system (1.1) will allow us to find a system of three conservation laws, whose unknowns are related in a natural way to a periodic solution to (1.1). The link with the analysis of the stability of periodic waves will be described in last section.

Let $\epsilon > 0$ be a small parameter and rescale the equations of (1.1) by means of the following change of variables $(x, t) \to (\epsilon x, \epsilon t)$.

Then the equations of (1.1) become

$$\begin{cases} v_t = u_x, \\ u_t + (\epsilon^2 v_{xx} + p)_x = 0, \\ p_t + a^2 u_x = \dfrac{1}{\epsilon}(\pi(v) - p). \end{cases} \tag{3.1}$$

We look for a homogenized system whose equations describe the mean behaviour of the solution to (1.1). To this aim, we ask that the solution to (3.1) $U = (v,u,p)^t$ satisfies the following asymptotic expansion in terms of powers of ϵ

$$U^\epsilon(x,t) = U^0(x,t,x/\epsilon) + \epsilon U^1(x,t,x/\epsilon) + ... \tag{3.2}$$

Denote by y the fast variable x/ϵ and assume that the functions which appear in the r.h.s. of the previous expansion are sufficiently regular. In accordance with the result of Lemma 1.2, we fix $U^0 = (v_0, u_0, \pi(v_0))^t$, where $(v_0, u_0, \pi(v_0))^t$ is the periodic solution to (1.1) of period X_0 and we ask that the functions $U^i, i \geq 1$, are periodic in y of period X_0. Notice that in our case u_0 depends only on (x,t), due to the result of (2.9). The result of Theorem 5.2, which we shall state in last section, will prove that our choice for the asymptotic expansion (3.2) is significant for our model. In addition, let us remark that the phase function, which we have chosen to define the fast variable, does not depend on t. As it will be clear below in performing the asymptotic analysis, we have fixed the phase function in such a way that it does not depend on t because the system (1.1) admits only periodic stationary TWS.

Let us substitute (3.2) in the equations of system (3.1) and identify the powers of ϵ. At order ϵ^{-1}, we find the only equation

$$(\pi(v_0) + (v_0)_{yy})_y = 0; \tag{3.3}$$

which is satisfied due to our assumptions. At order ϵ^0, we find the following system

$$\begin{cases} (v_0)_t = (u_0)_x + (u^1)_y, \\ (u_0)_t + 3(v_0)_{xyy} + 3(v^1)_{yyy} + (p_0)_x + (p^1)_y = 0, \\ (\pi(v_0))_t + a^2(u_0)_x + a^2(u^1)_y = \pi'(v_0)v^1 - p^1. \end{cases} \tag{3.4}$$

Set $< U^0 >= \frac{1}{X_0}\int_0^{X_0} U^0(x,t,y)dy$. In taking the mean over the period X_0 in the equations of (3.4), we obtain the following system

$$\begin{cases} \partial_t < v_0 >= \partial_x < u_0 >, \\ \partial_t < u_0 >= -\partial_x < \pi(v_0) >, \\ \partial_t < \pi(v_0) >= -a^2\partial_x < u_0 > + \dfrac{1}{X_0}\displaystyle\int_0^{X_0} [v^1\pi'(v_0) - p^1]dy, \end{cases} \tag{3.5}$$

where $< u_0 >= u_0(x,t)$.

Let us remark that, as a difference from the viscous case treated in [5], by means of the slow modulation of the periodic waves, we get from (1.1) only two conservation laws in the unknowns $< v_0 >, < u_0 >$. This fact is natural because of the presence of the relaxation term. It will be clear in last section when proving the main result, that we need an extra conservation law. Hence, we consider the equation $\tilde{\eta}_t + \tilde{q}_x - r = 0$, as found in the proof of Lemma 1.1 and plug the asymptotic expansion (3.2) into this identity. By following the same procedure as for the computation of (3.3) and (3.4), we find after averaging

$$\partial_t[a^2 < u_0^2 > + < \pi(v_0)^2 > + < h(a^2 v_0 + \pi(v_0)) > + a^2 < (v_{0y})^2 >] \\ + \partial_x[2au_0 < \pi(v_0) >] = 0. \quad (3.6)$$

Thus, the previous equation together with the conservation laws in system (3.5) allows us to obtain a first-order system of three conservation laws

$$\begin{cases} \partial_t < v_0 > - \partial_x < u_0 >= 0, \\ \partial_t < u_0 > + \partial_x < \pi(v_0) >= 0, \\ \partial_t[a^2 < u_0^2 > \\ + < \pi(v_0)^2 > + < h(a^2 v_0 + \pi(v_0)) > + a^2 < (v_{0y})^2 >] \\ + \partial_x[2au_0 < \pi(v_0) >] = 0. \end{cases} \quad (3.7)$$

In Section 5, we shall explain the link between the leading order terms of the Evans function in a neighbourhood of the origin and the determinant of a suitable matrix involving the equations (3.7).

4. Analysis of the Evans Function

In this section we shall investigate the spectral stability of a fixed periodic solution U^0 to (1.1), by studying the spectrum of the linearized equations about the periodic wave U^0:

$$\begin{cases} v_t = u_x, \\ u_t + (v_{xx} + p)_x = 0, \\ p_t + a^2 u_x = \pi'(v_0)v - p. \end{cases} \quad (4.1)$$

Since the previous system has periodic coefficients, the spectral analysis will be carried out by means of the Floquet's theory for linear differential operators with periodic coefficients. Let us recall some basic notions. Let L be a linear differential operator with $X-$periodic coefficients and domain $L^2(\mathcal{R})$. Let $L\omega = \lambda\omega$ be the associated eigenvalue problem. Due to the choice of the domain, the spectrum of the operator L is continuous (if we had chosen the space $L^2(\mathcal{R}/X\mathcal{Z})$ as domain of the operator, the spectrum would have been discrete). In order to study the spectrum, it is convenient to write the eigenvalue problem as a first order system and to consider the associated monodromy matrix $M(X, \lambda)$, satisfying $M(0, \lambda) = I_n$. It holds true that $\lambda \in \sigma(L)$ if and only if $\det(M(X, \lambda) - e^{i\theta} I_n) = 0$, for some real value θ. In other words, $\lambda \in \sigma(L)$ if $M(X, \lambda)$ has an eigenvalue on the unit circle S^1. Thanks to an argument due to Gardner [1], one can rewrite the previous determinant in equivalent form by choosing a basis in the kernel of $L - \lambda$.

Let us follow the procedure described above to investigate the stability of the periodic solution U^0 with period X_0, by means of the linearized system (4.1).

Consider the eigenvalue problem for the operator L_0 defined by the equations (4.1):

$$\begin{cases} u_x = \lambda v, \\ (v_{xx} + p)_x = -\lambda u, \\ a^2 u_x - \pi'(v_0)v + p = -\lambda p. \end{cases} \tag{4.2}$$

The domain of L_0 is $L^2(\mathcal{R})$.

Let us give the following definition.

Definition 4.1. The solution U^0 is said to be weakly spectrally stable if each element of the spectrum of L_0 is of non positive real part. The solution is spectrally unstable otherwise.

Let us write the system (4.2) as a first order system and set $Y = (u, v, v_x, v_{xx})^t$. After eliminating p, we obtain

$$\begin{pmatrix} u \\ v \\ v_x \\ v_{xx} \end{pmatrix}_x = \begin{pmatrix} \lambda v \\ v_x \\ v_{xx} \\ -\lambda u - \dfrac{1}{\lambda+1}(\pi'(v_0)v - a^2\lambda v)_x \end{pmatrix}. \tag{4.3}$$

We look now for a basis in the kernel of $L_0 - \lambda$. We derive the following linear homogeneous fourth order differential equation

$$(\lambda+1)u^{(4)} + (\pi'(v_0) - a^2\lambda)u^{(2)} + \pi''(v_0)v_0' u^{(1)} = -\lambda^2(\lambda+1)u. \tag{4.4}$$

Let us denote by $u_j(\cdot,\lambda)$, with $j \in \{1,2,3,4\}$, the functions, which represent a basis in the linear space of the solutions to (4.4).

Consider the initial value problem for the system (4.3) and let $M(\cdot,\lambda)$ be the associated monodromy operator. Therefore $Y(X_0) = M(X_0,\lambda)Y(0)$. If $Y_j(0,\lambda)$, $(j = 1,2,3,4)$, is the canonical basis in the vector space $\mathcal{R}^4$, then $M(X_0,\lambda)$ equals the matrix whose columns are $Y_j(X_0,\lambda)$ for $j \in \{1,2,3,4\}$.

The following property holds true: there exists $\lambda \in \mathcal{C}$ such that $\det(M(X_0,\lambda) - e^{i\theta}I_4) = 0$, for some $\theta \in \mathcal{R}$, if and only if λ belongs to the spectrum of L_0. As a consequence of this equality, the eigenvalue problem (4.3) admits a non-trivial bounded solution Y, which satisfies $Y(x+X_0) = e^{i\theta}Y(x)$. The definition of the monodromy matrix yields $Y(x+kX_0) = e^{ik\theta}Y(x)$, with $k \in \mathcal{N}$. Thus, $L_0Y = \lambda Y$, for every $x \in \mathcal{R}$; where $\lambda \in \mathcal{C}$ is as defined above.

Let us choose now a basis of $\ker(L_0 - \lambda)$. Fix the vectors $Y_j(x,\lambda)$, as $Y_j = (u_j, v_j, v_j', v_j'')^t$, for $j \in \{1,2,3,4\}, x \in \mathcal{R}$ and $\lambda \in \mathcal{C}$; where u_j is a solution to (4.4) and $\lambda v_j = u_j'$. Set $[Y] = Y(X_0) - Y(0)$.

We define then the Evans function of the problem as the 4×4 determinant of the matrix whose columns are

$$\begin{pmatrix} [u_j] + (1-e^{i\theta})u_j(0) \\ [v_j] + (1-e^{i\theta})v_j(0) \\ [v_j'] + (1-e^{i\theta})v_j'(0) \\ [v_j''] + (1-e^{i\theta})v_j''(0) \end{pmatrix}, \tag{4.5}$$

for $j \in \{1,2,3,4\}$.

Denote by $D(\lambda,\theta)$ the Evans function. We want to investigate now the behaviour of the function $D(\cdot,\theta)$ at the origin $\lambda = 0$. To this aim, let us consider the Taylor expansion of $Y_j(x,\lambda)$ at $\lambda = 0$ and look for the leading order terms in the rows of the matrix (4.5). Denote by $\alpha_j(x), \beta_j(x), \gamma_j(x)$ the first three terms in the Taylor formula

$$Y_j(x,\lambda) = \alpha_j(x) + \lambda\beta_j(x) + \lambda^2\gamma_j(x) + \cdots .$$

For practical reasons, we choose $\alpha_1(x) = (0, v_0', v_0'', -\pi'(v_0)v_0')^t$, where v_0 is the periodic solution to (2.11). Notice that the vector $(0, v_0', \pi'(v_0)v_0')^t \in \ker(L_0)$ and α_1 satisfies the system (4.3) in the case where $\lambda = 0$.

Owing to a technical reason, which will be clear below in computing the leading order terms, add the second row of the matrix (4.5) multiplied by $\pi'(v_0(0))$ to the last row. Hence we obtain that $D(\lambda,\theta)$ equals the determinant of the matrix

$$\left(\begin{array}{c} [u_j] + (1 - e^{i\theta})u_j(0) \\ [v_j] + (1 - e^{i\theta})v_j(0) \\ [v_j'] + (1 - e^{i\theta})v_j'(0) \\ [v_j''] + \pi'(v_0(0))[v_j] + (1 - e^{i\theta})(v_j''(0) + v_j(0)\pi'(v_0(0))) \end{array}\right)_{j=1,2,3,4} . \tag{4.6}$$

Let us write the Taylor expansion for u_j and v_j as follows

$$\begin{array}{c} u_j(x,\lambda) = \alpha_j^{(1)}(x) + \lambda\beta_j^{(1)}(x) + \lambda^2\gamma_j^{(1)}(x) + \cdots ; \\ v_j(x,\lambda) = \alpha_j^{(2)}(x) + \lambda\beta_j^{(2)}(x) + \lambda^2\gamma_j^{(2)}(x) + \cdots ; \end{array} \tag{4.7}$$

for every $j = 1, 2, 3, 4$. Since $u_j' = \lambda v_j$, we have the following identities

$$\alpha_j^{(1)'} = 0, \beta_j^{(1)'} = \alpha_j^{(2)} \quad \text{and} \quad \gamma_j^{(1)'} = \beta_j^{(2)},$$

for every $j \in \{1, 2, 3, 4\}$.

Thus we consider the leading order terms in the first row of (4.6), i.e. $[u_j] + (1 - e^{i\theta})u_j(0)$.

We have to study separately the cases $j = 1$ and $j > 1$. If $j = 1$, since $\alpha_1^{(1)} = 0$, $\beta_1^{(1)'} = v_0'$ and $\gamma_1^{(1)'} = \beta_1^{(2)}$, we get $[\alpha_1^{(1)}] = [\beta_1^{(1)}] = 0$ and $[\gamma_1^{(1)}] = \int_0^{X_0} \beta_1^{(2)}(x)dx$. Thus we obtain that the leading order term is a homogeneous polynomial of degree 2 in λ and θ:

$$\lambda^2 \int_0^{X_0} \beta_1^{(2)}(x)dx - i\theta\lambda\beta_1^{(1)}(0).$$

For $j > 1$, we get $\lambda[\beta_j^{(1)}] - i\theta\alpha_j^{(1)}(0)$.

As far as the second row $[v_j] + (1 - e^{i\theta})v_j(0)$ is concerned, if $j = 1$ then the leading order term is given by $\lambda[\beta_1^{(2)}] - i\theta v_0'(0)$; if $j > 1$, we have instead $[\alpha_j^{(2)}]$.

Similarly we find for the third row $[v_j'] + (1 - e^{i\theta})v_j'(0)$ that if $j > 1$, then the leading order term is $[\alpha_j^{(2)'}]$. In the case where $j = 1$, we get $\lambda[\beta_1^{(2)'}] - i\theta v_0''(0)$.

Consider now the last row

$$[v_j''] + \pi'(v_0(0))[v_j] + (1 - e^{i\theta})(v_j''(0) + v_j(0)\pi'(v_0(0))).$$

From the fourth equation in (4.3), we deduce

$$(\lambda + 1)v_j''' = -\lambda^2 u_j - \lambda u_j + a^2\lambda v_j' - (\pi'(v_0)v_j)'.$$

By means of the Taylor formulas for u_j and v_j, we obtain the following ordinary differential equations

$$\begin{aligned} &\alpha_j^{(2)'''} = -(\pi'(v_0)\alpha_j^{(2)})';\\ &\alpha_j^{(2)'''} + \beta_j^{(2)'''} = -\alpha_j^{(1)} + a^2\alpha_j^{(2)'} - (\pi'(v_0)\beta_j^{(2)})';\\ &\beta_j^{(2)'''} + \gamma_j^{(2)'''} = -\alpha_j^{(1)} - \beta_j^{(1)} + a^2\beta_j^{(2)'} - (\pi'(v_0)\gamma_j^{(2)})'; \end{aligned} \tag{4.8}$$

for every $j = 1, 2, 3, 4$.

As a consequence of the previous equations, we obtain

$$\begin{aligned} &[\alpha_j^{(2)''}] = -[\pi'(v_0)\alpha_j^{(2)}];\\ &[\beta_1^{(2)''}] = -[\pi'(v_0)\beta_1^{(2)}];\\ &[\beta_j^{(2)''}] = [\pi'(v_0)\alpha_j^{(2)}] + a^2[\alpha_j^{(2)}] - [\pi'(v_0)\beta_j^{(2)}] - X_0\alpha_j^{(1)};\\ &[\gamma_j^{(2)''}] + [\pi'(v_0)\gamma_j^{(2)}] = (\pi'(v_0(0)) + a^2)[\beta_j^{(2)}]\\ &-(\pi'(v_0(0)) + a^2)[\alpha_j^{(2)}] - \textstyle\int_0^{X_0} \beta_j^{(1)} dx; \end{aligned} \tag{4.9}$$

for every $j = 1, 2, 3, 4$.

Thus, we get that the leading order term in the last row of (4.6) in the case where $j = 1$ is given by

$$\lambda^2(\pi'(v_0(0)) + a^2)[\beta_1^{(2)}] - \lambda^2 \int_0^{X_0} v_0(x)dx \\ - i\theta\lambda(\beta_1^{(2)''}(0) + \pi'(v_0(0))\beta_1^{(2)}(0)).$$

For $j \geq 2$, we find

$$\lambda([\beta_j^{(2)''}] + [\pi'(v_0)\beta_j^{(2)}]) - i\theta(\alpha_j^{(2)''}(0) + \pi'(v_0(0))\alpha_j^{(2)}(0)).$$

Hence we may write the matrix whose elements are the leading order terms in a neighbourhood of the origin $\lambda = 0$.

The first column is given by

$$\begin{pmatrix} \lambda^2 \int_0^{X_0} \beta_1^{(2)}(x)dx - i\theta\lambda\beta_1^{(1)}(0) \\ \lambda[\beta_1^{(2)}] - i\theta v_0'(0) \\ \lambda[\beta_1^{(2)'}] - i\theta v_0''(0) \\ \lambda^2(\pi'(v_0(0)) + a^2)[\beta_1^{(2)}] - \lambda^2 \int_0^{X_0} v_0(x)dx - i\theta\lambda(\beta_1^{(2)''}(0) + \pi'(v_0(0))\beta_1^{(2)}(0)) \end{pmatrix}; \tag{4.10}$$

for $j > 1$, the columns of the matrix are

$$\begin{pmatrix} \lambda\int_0^{X_0}\alpha_j^{(2)}dx - i\theta\alpha_j^{(1)} \\ [\alpha_j^{(2)}] \\ [\alpha_j^{(2)\prime}] \\ \lambda(\pi'(v_0(0))+a^2)[\alpha_j^{(2)}] - \lambda X_0\alpha_j^{(1)} - i\theta(\alpha_j^{(2)\prime\prime}(0)+\pi'(v_0(0))\alpha_j^{(2)}(0)) \end{pmatrix}. \tag{4.11}$$

Let us denote by $\mathcal{A}$ the matrix whose columns are the previous ones.

Consider the following third order linear homogeneous differential equation

$$(z'' + \pi'(v_0)z)' = 0.$$

Because of (4.8), the functions $\alpha_j^{(2)}$ have to satisfy the previous equation for every $j \in \{1,2,3,4\}$.

In the case where $\lambda = 0$, the basis of vectors $Y = (u_j, v_j, v_j', v_j'')^t$, for $j = 1,2,3,4$ has to be chosen in such a way that the functions u_j and v_j satisfy the following system

$$\begin{cases} u_j' = 0, \\ (v_j'' + \pi'(v_0)v_j)' = 0. \end{cases} \tag{4.12}$$

Thus we fix $v_0', \alpha_2^{(2)}, \alpha_3^{(2)}$ as linearly independent solutions of the third order differential equation in (4.12) and we define the following basis of vectors in the kernel of the first order linear operator defined by system (4.3)

$$\begin{aligned} &Y_1 = (0, v_0', v_0'', v_0''')^t; \\ &Y_j = (0, \alpha_j^{(2)}, \alpha_j^{(2)\prime}, \alpha_j^{(2)\prime\prime})^t, \text{for j= 2,3;} \\ &Y_4 = (1,0,0,0)^t. \end{aligned} \tag{4.13}$$

Hence we deduce that the fourth column of the matrix $\mathcal{A}$ is given by

$$\begin{pmatrix} -i\theta \\ 0 \\ 0 \\ -\lambda X_0 \end{pmatrix}. \tag{4.14}$$

Thus we may state the following result.

Proposition 4.1. *Consider the first order system (4.3) with periodic coefficients. If $D(\lambda,\theta)$ is the associated Evans function defined in (4.5), then in a neighbourhood of the origin $\lambda = 0$,*

$$D(\lambda,\theta) = \det(\mathcal{A}) + O(|\lambda|^4 + |\theta|^4),$$

where $\det(\mathcal{A})$ *is a homogeneous polynomial of degree 3 in λ and θ.*

We conclude by proving the main results in next section.

5. Main Theorems

In the present section we explain how the behaviour of the Evans function at leading order, in a neighbourhood of the origin $\lambda = 0$, allows us to prove that the fixed periodic wave is spectrally unstable. Moreover, we state our main result which deals with a description of the zero set of the Evans function around the origin. This set is described in terms of the macroscopic variables of the first order conservation laws derived in Section 3 in order to characterize the slow modulation of the periodic stationary travelling waves.

Theorem 5.1. *Consider the equations of system (1.1). If* (u_0, v_0, p_0) *is a non trivial stationary periodic solution to (1.1), then* (u_0, v_0, p_0) *is spectrally unstable.*

Proof. Let us choose the functions $\alpha_2^{(2)}$ and $\alpha_3^{(2)}$ as linearly independent solutions of (4.12) in order to satisfy the following property:

$$\int_0^{X_0} \alpha_j^{(2)} = \alpha_j^{(2)''}(0) + \pi'(v_0(0))\alpha_j^{(2)}(0); \tag{5.1}$$

in the case where $j = 2, 3$.

Moreover, fix the vector $Y_1 = (u_1, v_1, v_1', v_1'')$ in such a way that the following relations hold true

$$\begin{aligned} &\beta_1^{(1)}(0) = \frac{1}{X_0}\int_0^{X_0} v_0(x)dx, \\ &\int_0^{X_0} \beta_1^{(2)}(x)dx = \beta_1^{(2)''}(0) + \pi'(v_0(0))\beta_1^{(2)}(0) - (\pi'(v_0(0)) + a^2)v_0'(0). \end{aligned} \tag{5.2}$$

Let us denote by μ_i, with $i = 0, 1, 2, 3$, the coefficients of the third order homogeneous polynomial, which represents $\det(\mathcal{A})$. We get

$$\det(\mathcal{A}) = \mu_3 X_0 \lambda^3 - \mu_2 X_0 i\theta\lambda^2 + \mu_1\theta^2\lambda - \mu_0 i\theta^3.$$

As a consequence of the previous assumptions, we obtain $\mu_3 = -\mu_1$ and $\mu_2 = -\mu_0$.

Therefore, $\det(\mathcal{A}) = (\mu_3\lambda + i\theta\mu_0)(\lambda^2 X_0 - \theta^2)$.

Thus the equation $\det(\mathcal{A}) = 0$ admits a solution λ of positive real part. By applying the result of Rouchè's Theorem on the persistence of zeros of holomorphic functions, we establish that the Evans function has a zero of positive real part. Therefore, in accordance with Definition 3.1, the fixed periodic wave of (1.1) is spectrally unstable.

By following the procedure of [5], we investigate now the link between the system of conservation laws (3.7) in the macroscopic variables and the matrix $\mathcal{A}$. After taking the sum of last two equations of (3.7), in view of the specific form of the density and the flux of these laws, we define the following vector fields

$$\begin{aligned} M &= \left(\frac{1}{X} \int_0^{X_0} v(x)dx, \frac{1}{X} \int_0^{X_0} (v'(x) + u_0)dx \right), \\ Q &= (u_0, -v'' - \pi'(v_0)u_0 v), \end{aligned} \tag{5.3}$$

where X is the period of a periodic wave. □

Set $\omega_0 = \frac{1}{X_0}$ and $M_0 = M(u_0, v_0)$. Let us prove now the main theorem.

Theorem 5.2. *Consider the system (1.1) and a fixed periodic wave (u_0, v_0, p_0). Let $\mathcal{A}$ be the matrix whose determinant represents the Evans function at leading order in a neighbourhood of the origin. If M and Q are the vector fields defined in (5.3), then the following result holds true:*

$$\begin{aligned} \det(\mathcal{A}) = &-(\pi'(v_0(0)) + a^2)\omega_0{}^{-2} \\ &\times \det\left(\lambda \frac{\partial(M + \omega_0 M_0 X)}{\partial \dot{V}}(U_0) + i\omega_0\theta \frac{\partial Q}{\partial \dot{V}}(U_0) \right). \end{aligned} \tag{5.4}$$

Proof. Let us define the following vectors in $\mathcal{R}^2$

$$\begin{aligned} &V_0'(0) = (-v_0'(0), -v_0''(0))^t; \\ &B = (\beta_1^2, \beta_1^{2\prime} - v_0(\pi'(v_0(0)) + a^2)^{-1})^t; \\ &K = (0, (\pi'(v_0(0)) + a^2)^{-1})^t; \\ &W_j = (\alpha_j^{(2)}, \alpha_j^{(2)\prime})^t,\ j = 2, 3. \end{aligned} \tag{5.5}$$

Consider the vectors $V_0'(0), [B], [W_j]$, with $j = 2, 3$. If they do not generate the vector space $\mathcal{R}^2$, then $D(\lambda, \theta) = 0$, for every λ and θ and the periodic wave is spectrally unstable. Assume that this is not the case. Hence $\mathcal{R}^2 =$Span$\{V_0'(0), [B], [W_2], [W_3]\}$.

Set $W = \delta_1 B + \delta_2 W_2 + \delta_3 W_3 + \delta_4 K$; where δ_i, with $i = 1, 2, 3, 4$ are real parameters. Moreover, denote by δ the vector $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ and by γ a real number. In relying on the techniques introduced in [5], define the functional $Z : \mathcal{R}^5 \to \mathcal{R}^2$, as follows

$$Z(\delta, \gamma) = \delta_1[B] + \delta_2[W_2] + \delta_3[W_3] + \gamma V_0'(0).$$

Let X be the period of a periodic wave solution to (1.1). Thus we deduce $dX(\delta,\gamma)=\gamma$.

Consider the vector fields M and Q defined in (5.3). Since u_0 is a real constant, assume that $u_0=1$. According to standard rules, $d(XM)=M_0dX+X_0dM$.

Let us compute the differentials $d(XM)(\delta,\gamma)$ and $dQ(\delta,\gamma)$. We obtain

$$d(XM)(\delta,\gamma)=\gamma V_0(0)+\delta_1\int_0^{X_0}Bdx+\delta_2\int_0^{X_0}W_2dx + \delta_3\int_0^{X_0}W_3dx+\delta_4\int_0^{X_0}Kdx. \quad (5.6)$$

As far as $dQ(\delta,\gamma)$ is concerned, let us compute $(\pi'(v_0(0))+a^2)^{-1}dQ(\delta,\gamma)$. We deduce

$$\begin{aligned}(\pi'(v_0(0))+a^2)^{-1}dQ(\delta,\gamma)\\ &=\sum_{j=2}^{3}\delta_j\left(0,(-\alpha_j^{(2)''}-\pi'(v_0)\alpha_j^{(2)})(\pi'(v_0(0))+a^2)^{-1}\right)\\ &+\delta_4\left((1,0)+\delta_1\left(0,(-\beta_1^{(2)''}-\pi'(v_0)\beta_1^{(2)})(\pi'(v_0(0))+a^2)^{-1}\right).\right.\end{aligned} \quad (5.7)$$

Consider now the restriction to $\ker Z$ of $\lambda d(M+\omega_0M_0X)+i\omega_0\theta dQ$ and compute the determinant of the matrix, which defines the following linear map

$$(\delta,\gamma)\longrightarrow\begin{pmatrix}\delta_1[B]+\delta_2[W_2]+\delta_3[W_3]+\gamma V_0'(0)\\ \lambda d(M+\omega_0M_0X)+i\omega_0\theta d\tilde{Q}\end{pmatrix}. \quad (5.8)$$

Parametrize as follows: $\delta_1=\lambda\rho,\delta_i=\delta_i$, with $i=2,3,4$, and $\gamma=i\theta\rho$.

By taking into account (5.8), we derive the corresponding matrix. The first column is given by

$$\begin{pmatrix}\lambda[B]+i\theta V_0'(0)\\ \lambda\omega_0 i\theta V_0(0)+\lambda^2\omega_0\int_0^{X_0}Bdx-\omega_0 i\theta(\beta_1^{(2)''}(0)+\pi'(v_0(0))\beta_1^{(2)}(0))(\pi'(v_0(0))+a^2)^{-1}\end{pmatrix}. \quad (5.9)$$

The second and third columns are

$$\begin{pmatrix}[W_j]\\ \lambda\omega_0\int_0^{X_0}W_jdx-\omega_0 i\theta(\alpha_j^{(2)''}(0)+\pi'(v_0(0))\alpha_j^{(2)}(0))(\pi'(v_0(0))+a^2)^{-1}\end{pmatrix}; \quad (5.10)$$

for $j=2,3$.

Finally, we obtain that the fourth column is given by

$$\begin{pmatrix} 0 \\ 0 \\ \omega_0 i\theta \\ \lambda(\pi'(v_0(0)) + a^2)^{-1}) \end{pmatrix}. \tag{5.11}$$

If v_0 is the fixed periodic wave to (1.1), the shifted functions $v_0(\cdot + h)$, for every $h \in \mathcal{R}$ are periodic solutions in the same class of equivalence. We may assume that $v_0'(0) = 0$; if $v_0'(0) \neq 0$, there exists an equivalent periodic solution, which satisfies this condition.

Let us choose now the function $\beta_1^{(1)}$: it holds true that $\beta_1^{(1)'} = v_0'$; thus we fix $\beta_1^{(1)}(0) = v_0(0)$. By taking into account the previous assumptions, we conclude the proof of the theorem.

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