International Journal of Pure and Applied Mathematics
Volume 22 No. 2 2005, 245-265

b-SEMI-E-CONVEX FUNCTIONS AND b-SEMI-E-CONVEX PROGRAMMING

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Abstract: In this paper, a new class of non-convex functions, called b-semi-E-convex (quasi-b-semi-E-convex, pseudo-$\bar{b}$-semi-E-convex) functions is introduced, and some of their basic characters are discussed. At the same time, some sufficient optimality conditions and duality theorems for the nonlinear b-semi-E-convex programming with inequality constraints are studied.

AMS Subject Classification: 26A33
Key Words: E-convex set, b-semi-E-convex function, b-semi-E-convex programming, optimality conditions, duality theorems

1. Introduction

Since convexity and generalized convexity plays a central role in optimization theory, the search on them becomes one of the most important aspects in mathematical programming. During the past several decades, various significant generalizations of convexity were presented (see [2], [10], [9], [5], [4], [6], [7], [3], [1]). Bector and Singh [2] brought forward a class of functions, called b-vex

Received: June 28, 2005

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functions and discussed their basic properties. Members of this class satisfy most of the basic properties of convex functions. After several years, Youness [10] introduced a class of sets and a class of functions, called E-convex sets and E-convex functions. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions. Later, Yang [9] and Jian [5] both rectified and supplemented Youness' work. By extending the E-convexity, Chen [4] and Jian [6] introduced the concepts of semi-E-convexity and (E, F)-convexity, respectively. More recently, Jian, Hu, Tang and Zheng [7] presented a class of semi-(E, F)-convex function and semi-(E, F)-convex programming, and discussed their basic properties.

In this paper, based on the definitions of b-vex and semi-E-convex functions, we present a class of b-semi-E-convex (quasi-b-semi-E-convex, pseudo-$\bar{b}$-semi-E-convex) functions and discuss b-semi-E-convex programming.

The paper is organized as follows: some preliminary concepts which have relationships with our work are stated in the next section. In Section 3, we introduce the definitions of b-semi-E-convex, quasi-b-semi-E-convex and pseudo-$\bar{b}$-semi-E-convex functions, then discuss their basic properties. In Section 4, we discuss b-semi-E-convex programming. In Section 5, we study sufficient optimality conditions. In Section 6, some duality results are also obtained by associating a Wolfe dual problem (see [1], [8]) with the generalized convex programming.

2. Preliminary Concepts

Definition 2.1. (see [2]) Let X be a nonempty convex subset in R^n, and let R_+ denotes the set of nonnegative real numbers. Let $f : X \to R$, $b_1 : X \times X \times [0,1] \to R_+$, $b_2 : X \times X \times [0,1] \to R_+$. The function f is said to be b-vex at point $x^0 \in X$ with respect to b_1 and b_2 if

$$f(\lambda x + (1-\lambda)x^0) \le b_1(x, x^0, \lambda) f(x) + b_2(x, x^0, \lambda) f(x^0), \quad \forall x \in X, \quad \forall \lambda \in [0,1], \quad (2.1)$$

with $b_1(x, x^0, \lambda) \ge 0$, $b_2(x, x^0, \lambda) \ge 0$, and $b_1(x, x^0, \lambda) + b_2(x, x^0, \lambda) = 1$. f is said to be b-vex on X if it is b-vex at each $x \in X$.

If one takes (see [3])

$$b_1(x, x^0, \lambda) = \lambda b(x, x^0, \lambda), \qquad b_2(x, x^0, \lambda) = 1 - \lambda b(x, x^0, \lambda),$$

where

$$b(x, x^0, \lambda) \ge 0, \quad \lambda b(x, x^0, \lambda) \in [0,1], \quad \forall \lambda \in [0,1],$$

then (2.1) can be completely described as follows:

$$f(\lambda x + (1-\lambda)x^0) \le \lambda b(x, x^0, \lambda) f(x) + (1 - \lambda b(x, x^0, \lambda)) f(x^0), \quad \forall x \in X, \quad \forall \lambda \in [0,1]. \quad (2.2)$$

Definition 2.2. (see [10]) A set $M \subset R^n$ is said to be E-convex if there is a map $E : M \to R^n$ such that

$$\lambda E(x) + (1-\lambda)E(y) \in M, \quad \forall x, y \in M, \quad \forall \lambda \in [0,1]. \quad (2.3)$$

Definition 2.3. (see [10]) A function $f : R^n \to R$ is said to be E-convex on a set $M \subset R^n$ if there is a map $E : R^n \to R^n$ such that M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(E(x)) + (1-\lambda) f(E(y)), \quad \forall x, y \in M, \quad \forall \lambda \in [0,1]. \quad (2.4)$$

Definition 2.4. (see [4]) A function $f : R^n \to R$ is said to be semi-E-convex on a set $M \subseteq R^n$ if there is a map $E : R^n \to R^n$ such that M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in M, \quad \forall \lambda \in [0,1]. \quad (2.5)$$

3. b-Semi-E-Convex Functions and Their Properties

In this section, we first present the definitions of b-semi-E-convex function, quasi-b-semi-E-convex function, pseudo-$\bar{b}$-semi-E-convex function, then discuss their main properties.

For convenience of discussion, We assume always that:

(i) $\emptyset \neq M \subseteq R^n$ is an E-convex set with respect to map $E : M \to R^n$. Of course, different problems have different set M and map E, resp.

(ii) R_+ denotes the set of nonnegative real numbers.

Definition 3.1. A function $f : M \to R$ is said to be b-semi-E-convex at point $y \in M$, if there exists a function $b(x, y, \lambda) : M \times M \times [0,1] \to R_+$, such that

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda b(x, y, \lambda) f(x) + (1 - \lambda b(x, y, \lambda)) f(y), \quad \forall x \in M, \forall \lambda \in [0,1], \quad (3.1)$$

with $\lambda b(x, y, \lambda) \in [0, 1]$. Moreover, f is said to be strictly b-semi-E-convex at point y if the inequality is strict for $x \neq y$ and $\lambda \in (0, 1)$. Furthermore, f is said to be (strictly) b-semi-E-convex on M if it is (strictly) b-semi-E-convex at each point $y \in M$, and f is said to be (strictly) b-semi-E-concave if $-f$ is (strictly) b-semi-E-convex.

Definition 3.2. A function $f : M \to R$ is said to be quasi-b-semi-E-convex at point $y \in M$ if there exists a function $b(x, y, \lambda) : M \times M \times [0, 1] \to R_+$ such that

$$f(x) \leq f(y) \Rightarrow b(x, y, \lambda) f(\lambda E(x) + (1 - \lambda)E(y)) \leq b(x, y, \lambda) f(y), \quad \forall x \in M, \quad \forall \lambda \in [0, 1]. \tag{3.2}$$

Function f is said to be quasi-b-semi-E-convex on M if it is quasi-b-semi-E-convex at each point $y \in M$.

Definition 3.3. A differentiable function $f : M \to R$ is said to be pseudo-$\bar{b}$-semi-E-convex at point $y \in M$ if there exists a function $\bar{b} : M \times M \to R_+$ such that

$$\nabla f(E(y))^T (E(x) - E(y)) \geq 0 \Rightarrow \bar{b}(x, y) f(x) \geq \bar{b}(x, y) f(y), \quad \forall x \in M. \tag{3.3}$$

Function f is said to be pseudo-$\bar{b}$-semi-E-convex on M if it is pseudo-$\bar{b}$-semi-E-convex at each point $y \in M$.

For convenience of writing, in the rest of this paper, we denote simply that

$$b_i \stackrel{\triangle}{=} b_i(x, y, \lambda), \quad i = 0, 1, 2, \ldots, \qquad b \stackrel{\triangle}{=} b(x, y, \lambda), \quad \forall x, y \in M, \quad \forall \lambda \in [0, 1].$$

The following propositions describe the relationships between b-semi-E-convexity (quasi-b-semi-E-convexity, pseudo-$\bar{b}$-semi-E-convexity) and some previous generalized convexities.

Proposition 3.1. *Each convex function f is a b-semi-E-convex function.*

Proof. This conclusion can be proved by taking maps $E(x) = x$ and $b(x, y, \lambda) \equiv 1$. □

Remark 3.1. The converse of Proposition 3.1 is not necessarily true. A counterexample is given as follows.

Example 3.1. Define functions $f : X = [-1, 1] \to R$ and $E : X \to R$, $b : X \times X \times [0, 1] \to R_+$ by

$$f(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq 0; \\ x + 1, & \text{if } \quad 0 < x < 1; \\ 1, & \text{if } \quad x = 1, \end{cases}$$

$$E(x) = -x^2, \quad b(x,y,\lambda) = \begin{cases} 1, & \text{if } \lambda = 0; \\ \frac{|xy|}{\lambda}, & \text{if } \lambda \neq 0, \end{cases} \quad \forall x, y \in X, \quad \forall \lambda \in [0,1].$$

Obviously, X is E-convex since

$$\lambda E(x) + (1-\lambda)E(y) = -(\lambda x^2 + (1-\lambda)y^2) \in [-1,0] \subset X, \quad \forall x, y \in X, \quad \forall \lambda \in [0,1].$$

To show f is b-semi-E-convex on X, firstly, we know that

$$b \geq 0, \;\; \lambda b = \begin{cases} 0, & \text{if } \lambda = 0; \\ |xy|, & \text{if } \lambda \neq 0, \end{cases} \quad \lambda b \in [0,1], \;\; \forall \lambda \in [0,1], \;\; \forall x, y \in X.$$

Next, it follows

$$f(\lambda E(x) + (1-\lambda)E(y)) = 0 \leq \lambda b f(x) + (1-\lambda b)f(y), \;\; \forall x, y \in X, \;\; \forall \lambda \in [0,1].$$

However, f is not convex on X because

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y), \quad \text{for } x = -\frac{1}{5}, \;\; y = \frac{4}{5}, \;\; \lambda = \frac{1}{2}.$$

Proposition 3.2. *Every b-vex function f is a b-semi-E-convex function with respect to map $E(x) = x$.*

Remark 3.2. The converse of Proposition 3.2 is not necessarily true.

The function f considered in Example 3.1 is b-semi-E-convex but not b-vex because

$$f(\lambda x + (1-\lambda)y) > \;\lambda b f(x) + (1-\lambda b)f(y), \quad \text{for } x = -\frac{1}{5}, \;\; y = 1, \;\; \lambda = \frac{1}{2}.$$

Proposition 3.3. *Each semi-E-convex function f is a b-semi-E-convex function with respect to map $b(x,y,\lambda) \equiv 1$.*

Remark 3.3. The converse of Proposition 3.3 is not necessarily true. It can be seen from the following example.

Example 3.2. Define functions $f : X = [0,1] \to R$ and $E : X \to R, \; b : X \times X \times [0,1] \to R_+$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 0; \\ 2, & \text{if } 0 < x \leq 1, \end{cases} \quad E(x) = x,$$

$$b(x,y,\lambda)=\begin{cases} 0, & \text{if } \ x=0; \\ \frac{1}{\lambda}, & \text{if } \ x\neq 0,\ y=0,\ \lambda\neq 0; \\ 1, & \text{otherwise,} \end{cases} \quad \forall x,y\in X, \quad \lambda\in[0,1].$$

It is easy to show that X is E-convex, and f is b-semi-E-convex on X. However, f is not semi-E-convex with respect to the same map E because

$$f(\lambda E(x)+(1-\lambda)E(y)) > \lambda f(x)+(1-\lambda)f(y), \quad \text{for } x=0,\ y=1,\ \lambda=\frac{1}{2}.$$

Proposition 3.4. *Let $f: M\to R$ be an E-convex function. If $f(E(x))\leq f(x)$, $\forall x\in M$, then f is a b-semi-E-convex function with respect to map $b(x,y,\lambda)\equiv 1$.*

Proof. Since $f: M\to R$ is an E-convex function on set M, taking into account the given conditions, for $\forall x,y\in M$, $\forall\lambda\in[0,1]$, we have

$$\begin{aligned} f(\lambda E(x)+(1-\lambda)E(y)) &\leq \lambda f(E(x))+(1-\lambda)f(E(y)) \\ &\leq \lambda f(x)+(1-\lambda)f(y)=\lambda b(x,y,\lambda)f(x)+(1-\lambda b(x,y,\lambda))f(y). \end{aligned}$$

Hence f is b-semi-E-convex with respect to map $b(x,y,\lambda)\equiv 1$ on M. □

Remark 3.4. Generally, a b-semi-E-convex function f is not an E-convex function.

Consider the function f in Example 3.2. Function f is b-semi-E-convex but not E-convex because

$$f(\lambda E(x)+(1-\lambda)E(y)) > \lambda f(E(x))+(1-\lambda)f(E(y)), \quad \text{for } x=0,\ y=1,\ \lambda=\frac{1}{2}.$$

Proposition 3.5. *Each b-semi-E-convex function on an E-convex set M is quasi-b-semi-E-convex on M.*

Proof. In fact, for b-semi-E-convex function f and $f(x)\leq f(y)$, $x,y\in M$ as well as $\lambda\in[0,1]$, one has

$$f(\lambda E(x)+(1-\lambda)E(y)) \leq \lambda b f(x)+(1-\lambda b)f(y)\leq f(y).$$

Since $b(x,y,\lambda)\geq 0$, so

$$b(x,y,\lambda)f(\lambda E(x)+(1-\lambda)E(y)) \leq b(x,y,\lambda)f(y).$$

This shows that f is quasi-b-semi-E-convex on M. □

Remark 3.5. The converse of Proposition 3.5 is not necessarily true. For showing this, a counterexample is given as follows.

Example 3.3. Define functions $f : X = [0, \frac{\pi}{2}] \to R$ and $E : X \to R,\ b : X \times X \times [0,1] \to R_+$ by

$$f(x) = \begin{cases} \sin x, & \text{if } \ 0 \le x < \frac{\pi}{2}; \\ \frac{1}{2}, & \text{if } \ x = \frac{\pi}{2}, \end{cases} \qquad E(x) = x,$$

$$b(x,y,\lambda) = \begin{cases} 0, & \text{if } \ x = \frac{\pi}{2} \text{ or } y = \frac{\pi}{2}; \\ 1, & \quad \text{otherwise}, \end{cases} \qquad \forall \lambda \in [0,1].$$

It is easy to show that X is E-convex, and f is quasi-b-semi-E-convex on X. f is not b-semi-E-convex since

$$f(\lambda E(x) + (1-\lambda)E(y)) > \lambda b f(x) + (1-\lambda b) f(y), \quad \text{for } x = 0,\ y = \frac{\pi}{2},\ \lambda = \frac{1}{2}.$$

Proposition 3.6. *Suppose that f is a b-semi-E-convex function at point $y \in M$. If $f(E(y)) \ge f(y)$, and $\bar{b}(x,y) = \lim_{\lambda \to 0^+} b(x,y,\lambda),\ \forall x \in M$, then f is pseudo-$\bar{b}$-semi-E-convex at point $y \in M$ with respect to $\bar{b}(x,y)$.*

Proof. Since f is b-semi-E-convex function at point $y \in M$, $f(E(y)) \ge f(y)$, combining Taylor expansion, one has

$$\begin{aligned} & f(\lambda E(x) + (1-\lambda)E(y)) = f(E(y) + \lambda(E(x) - E(y))) \\ & \quad = f(E(y)) + \lambda \nabla f(E(y))^T (E(x) - E(y)) + o(\lambda) \le \lambda b f(x) + (1-\lambda b) f(y) \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad = f(y) + \lambda b (f(x) - f(y)). \end{aligned}$$

Taking into account $f(E(y)) \ge f(y)$, we further have

$$\lambda \nabla f(E(y))^T (E(x) - E(y)) + o(\lambda) \le \lambda b (f(x) - f(y)).$$

Dividing the inequality above by λ and taking $\lambda \to 0^+$, we have

$$\nabla f(E(y))^T (E(x) - E(y)) \le \bar{b}(x,y)(f(x) - f(y)).$$

If $\nabla f(E(y))^T (E(x) - E(y)) \ge 0,\ \forall x \in M$, then $\bar{b}(x,y) f(x) \ge \bar{b}(x,y) f(y)$. It shows f is pseudo-$\bar{b}$-semi-E-convex at point $y \in M$. ☐

Remark 3.6. The converse of Proposition 3.6 is not necessarily true. A counterexample is given as follows.

Example 3.4. Define functions $f : X = [0,1] \to R$ and $E : X \to R,\ \bar{b} : X \times X \to R_+$ by

$$f(x) = x, \qquad E(x) = \sqrt{x}, \qquad \bar{b}(x,y) \equiv 1.$$

One can easily show that X is E-convex, and f is pseudo-$\bar{b}$-semi-E-convex on X. Consider points $x = 0$, $y = \frac{1}{4}$ and $\lambda = \frac{1}{4}$, one has

$$f(\lambda E(x) + (1-\lambda)E(y)) = \frac{3}{8}, \quad \lambda b f(x) + (1-\lambda b) f(y) = \frac{1}{4} - \frac{1}{16} b.$$

So, $f(\lambda E(x) + (1-\lambda)E(y)) > \lambda b f(x) + (1-\lambda b) f(y)$ is always hold for any map $b : X \times X \times [0,1] \to R_+$ at points $x = 0$, $y = \frac{1}{4}$ and $\lambda = \frac{1}{4}$. Therefore f is not b-semi-E-convex with respect to any map b.

In the rest of this section, we will discuss some main properties of b-semi-E-convex functions.

Proposition 3.7. *If function $f_i : R^n \to R$ is b_i-semi-E-convex on M with respect to b_i for each $i \in I = \{1, 2, \ldots, m\}$, then the function $f(x) = \max\ \{f_i(x) : i \in I\}$ is b_f-semi-E-convex on M with respect to b_f defined as $b_f = b_f(x, y, \lambda) = b_t(x, y, \lambda)$, where $t = t(x, y, \lambda) = \min\ \{j : j \in \operatorname{argmax}\{\ f_i\ (\lambda E(x) + (1-\lambda)E(y)) : i \in I\}\}$.*

Proof. Since f_i is b_i-semi-E-convex on the E-convex set M, for each $x, y \in M$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)E(y)) &= \max\ \{f_i(\lambda E(x) + (1-\lambda)E(y)) : i \in I\} \\ &= f_t(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda b_t f_t(x) + (1-\lambda b_t) f_t(y) \\ &\leq \lambda b_f f(x) + (1-\lambda b_f) f(y). \end{aligned}$$

So, the function $f(x) = \max\ \{f_i(x) : i \in I)\}$ is b_f-semi-E-convex on M. □

Proposition 3.8. *If functions $f_i : R^n \to R$ is b-semi-E-convex and bounded on M for each $i \in I$ (may be infinite), then the function $f(x) = \sup_{i \in I}\ \{f_i(x)\}$ is b-semi-E-convex on M.*

Proof. Since f_i is b-semi-E-convex and bounded on the E-convex set M, for any $x, y \in M$ and $\lambda \in [0, 1]$, one has

$$\begin{aligned} & f(\lambda E(x) + (1-\lambda)E(y)) = \sup_{i \in I}\ \{f_i(\lambda E(x) + (1-\lambda)E(y))\} \\ & \leq \sup_{i \in I}\ \{\lambda b f_i(x) + (1-\lambda b) f_i(y)\} \leq \lambda\ b\ \sup_{i \in I}\ \{f_i(x)\} + (1-\lambda b)\ \sup_{i \in I}\ \{f_i(y)\} \\ & = \lambda b f(x) + (1-\lambda b) f(y). \end{aligned}$$

So, the function $f(x) = \sup_{i \in I}\ \{f_i(x)\}$ is b-semi-E-convex on M. □

Proposition 3.9. *If functions $f_i : R^n \to R$ is b-semi-E-convex on M ($i = 1, 2, \ldots, k$), then $h(x) = \sum_{i=1}^{k} a_i f_i(x)$ ($a_i \geq 0$, $i = 1, 2, \ldots, k$) is a b-semi-E-convex function on M.*

Proof. Since f_i is b-semi-E-convex on the E-convex set M, we get

$$f_i(\lambda E(x) + (1-\lambda)E(y)) \le \lambda b f_i(x) + (1-\lambda b) f_i(y), \quad \forall x, y \in M,\ \forall \lambda \in [0,1].$$

So

$$h(\lambda E(x) + (1-\lambda)E(y)) = \sum_{i=1}^{k} a_i f_i(\lambda E(x) + (1-\lambda)E(y))$$
$$\le \sum_{i=1}^{k} a_i(\lambda b f_i(x) + (1-\lambda b) f_i(y)) = \lambda b \sum_{i=1}^{k} a_i f_i(y) + (1-\lambda b) \sum_{i=1}^{k} a_i f_i(y)$$
$$= \lambda b h(x) + (1-\lambda b) h(y).$$

Therefore, the function $f(x) = \sum_{i=1}^{k} a_i f_i(x)$ is b-semi-E-convex on M. □

Proposition 3.10. *If function $f : R^n \to R$ is b-semi-E-convex on M, and $\phi : R \to R$ is a nondecreasing convex function, then the composite function $\phi(f)$ is b-semi-E-convex on M.*

Proof. Since f is b-semi-E-convex on the E-convex set M, one has

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda b f(x) + (1-\lambda b) f(y), \quad \forall x, y \in M, \quad \forall \lambda \in [0,1].$$

In view of $\phi : R \to R$ being nondecreasing convex function, we get

$$\phi(f(\lambda E(x) + (1-\lambda)E(y))) \le \phi(\lambda b f(x) + (1-\lambda b) f(y))$$
$$\le \lambda b \phi(f(x)) + (1-\lambda b)\phi(f(y)).$$

This shows that $\phi(f)$ is b-semi-E-convex on M. □

Proposition 3.11. *Assume that function $f : R^n \to R$ is b-semi-E-convex on M. Then the level set*

$$S_\alpha = \{x : x \in M,\ f(x) \le \alpha\}$$

is an E-convex set for each $\alpha \in R$.

Proof. Assume that $x, y \in S_\alpha$ and $\lambda \in [0,1]$, from the given conditions, we have

$$f(x) \le \alpha, \quad f(y) \le \alpha, \quad \lambda E(x) + (1-\lambda)E(y) \in M,$$
$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda b f(x) + (1-\lambda b) f(y) \le \alpha.$$

So $\lambda E(x) + (1-\lambda)E(y) \in S_\alpha$. This follows that the set S_α is E-convex. □

Remark 3.7. The converse of Proposition 3.11 is not necessarily true. For showing this, a counterexample is given as follows.

Example 3.5. Define functions $f : X = [-\pi,\ \pi] \to R$ and $E : X \to R$ by

$$f(x) = \begin{cases} x + \frac{\pi}{2}, & \text{if} \quad -\pi \le x < -\frac{\pi}{2}; \\ \cos x, & \text{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}; \\ 2, & \text{if} \quad \frac{\pi}{2} < x \le \pi, \end{cases}$$

$$E(x) = \begin{cases} -\pi, & \text{if} \quad -\pi \le x \le \frac{\pi}{2}; \\ x, & \text{if} \quad \frac{\pi}{2} < x \le \pi. \end{cases}$$

Obviously, the set X is an E-convex set, and the level set

$$S_\alpha = \begin{cases} \emptyset, & \text{if} \quad \alpha < -\frac{\pi}{2}; \\ [-\pi,\ \alpha - \frac{\pi}{2}], & \text{if} \quad -\frac{\pi}{2} \le \alpha < 0; \\ [-\pi, -\arccos\alpha] \cup [\arccos\alpha, \frac{\pi}{2}], & \text{if} \quad 0 \le \alpha < 1; \\ [-\pi,\ \frac{\pi}{2}], & \text{if} \quad 1 \le \alpha < 2; \\ [-\pi,\ \pi], & \text{if} \quad \alpha \ge 2 \end{cases}$$

is always E-convex. Consider points $x = -\pi,\ y = \pi$ and $\lambda = \frac{1}{8}$, one has

$$f(\lambda E(x) + (1-\lambda)E(y)) = f(\frac{1}{8} \times (-\pi) + \frac{7}{8}\pi) = f(\frac{3\pi}{4}) = 2,$$

and

$$\lambda b f(x) + (1 - \lambda b) f(y) = \frac{1}{8} b f(-\pi) + (1 - \frac{1}{8} b) f(\pi) = 2 - \frac{1}{8}(\frac{\pi}{2} + 2)b.$$

So, for any map $b : X \times X \times [0,1] \to R_+$, satisfying $0 < b = b(-\pi, \pi, \frac{1}{8})$, we have

$$f(\lambda E(x) + (1-\lambda)E(y)) > \lambda b f(x) + (1 - \lambda b) f(y),$$
$$\text{for } x = -\pi,\ y = \pi \text{ and } \lambda = \frac{1}{8}.$$

Thus the function $f(x)$ is not b-semi-E-convex on the E-convex set X with respect to any map $b : X \times X \times [0,1] \to R_+$ satisfying $b(-\pi, \pi, \frac{1}{8}) > 0$.

4. b-Semi-E-Convex Programming

In this section, we consider the following nonlinear programming problem

$$(P) \qquad \min \{f(x) \mid x \in M \subset R^n\}.$$

Definition 4.1. The program (P) is said to be a b-semi-E-convex programming if there exist maps $E : M \to R$ and $b : M \times M \times [0, 1] \to R_+$ such that the feasible set M is E-convex and the objective function f is b-semi-E-convex on M.

Theorem 4.1. *The set of optimal solutions of the b-semi-E-convex programming (P) is an E-convex set.*

Proof. Let $\bar{x}$ be an optimal solution of (P) and $\alpha = f(\bar{x})$, then the optimal solution set Ω can be expressed as: $\Omega = \{x \in M : f(x) \leq \alpha\} = S_\alpha$. So $\Omega = S_\alpha$ is an E-convex set by Proposition 3.11. □

Theorem 4.2. *Suppose that x^* is a local optimal solution of the b-semi-E-convex programming (P). If $E(x^*) = x^*$, and $b(y, x^*, \lambda) > 0$ holds for any $y \in M$ and $\lambda > 0$ sufficient small, then x^* is a global optimal solution of (P).*

Proof. By contradiction, suppose that x^* is not a global optimal solution of (P), so $f(\bar{y}) < f(x^*)$ for some $\bar{y} \in M$. Since $E(x^*) = x^*$ and f is b-semi-E-convex on the E-convex set M, for $\lambda > 0$ small enough, $b(\bar{y}, x^*, \lambda) > 0$ and $x^* + \lambda(E(\bar{y}) - x^*) = \lambda E(\bar{y}) + (1 - \lambda)E(x^*) \in M$, we have

$$\begin{aligned} f(x^* + \lambda(E(\bar{y}) - x^*)) &= f(\lambda E(\bar{y}) + (1 - \lambda)E(x^*)) \\ &\leq \lambda b(\bar{y}, x^*, \lambda) f(\bar{y}) + (1 - \lambda b(\bar{y}, x^*, \lambda)) f(x^*) < f(x^*). \end{aligned}$$

Hence, this strict inequality contradicts the fact that x^* is a local optimal solution, so x^* is a global optimal solution of (P). □

Theorem 4.3. *Suppose that function $f : M \to R$ is strictly b-semi-E-convex on an E-convex M. Then the global optimal solution of b-semi-E-convex programming (P) is unique.*

Proof. By contradiction, suppose that $x^1, x^2 \in M$ are two global optimal solutions of (P) and $x^1 \neq x^2$, since f is strictly b-semi-E-convex on the E-convex set M, we have

$$x^1, x^2 \in M, \quad f(x^1) = f(x^2), \quad \lambda E(x^2) + (1 - \lambda)E(x^1) \in M, \quad \forall \lambda \in [0, 1],$$

and for all $\lambda \in (0, 1)$,

$$f(\lambda E(x^2)+(1-\lambda)E(x^1)) < \lambda b(x^2,x^1,\lambda)f(x^2)+(1-\lambda b(x^2,x^1,\lambda))f(x^1) = f(x^1).$$

This contradicts that x^1 is a global solution of (P). Hence the global solution of (P) is unique. □

In the subsequent discussion of this section, we apply the associated results above to nonlinear programming problems with inequality constraints as follows:

$$(P_g) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x)\le 0, \quad i\in I=\{1,2,...,m\}, \\ & x\in R^n. \end{array}$$

Denote the feasible set of (P_g) by $M_g=\{\ x\in R^n \mid g_i(x)\le 0, \quad i\in I\}$.

Theorem 4.4. *Suppose that there exist maps $E: R^n\to R$, b, $b_i(i\in I): R^n\times R^n\times[0,1]\to R_+$ such that f and g_i $(i\in I)$ are b_i-semi-E-convex functions on R^n with respect to maps b and b_i. Then:*

(i) the feasible set M_g of problem (P_g) is an E-convex set. Thereby, (P_g) is a b-semi-E-convex programming;

(ii) the optimal solution set of (P_g) is E-convex;

(iii) let x^* be a local optimal solution of (P_g), if $E(x^*)=x^*$, and $b(y,x^*,\lambda)>0$ holds for any $y\in M$ and $\lambda>0$ sufficient small, then x^* is a global optimal solution of (P_g);

(iv) if f is strictly b-semi-E-convex, then the global optimal solution of (P_g) is unique.

Proof. (i) Since g_i is b_i-semi-E-convex on R^n for each $i\in I$ and $M_g\subseteq R^n$, for $\forall x^1,x^2\in M_g$, $\forall\lambda\in[0,1]$, one has

$$\lambda E(x^1)+(1-\lambda)E(x^2)\in R^n, \quad g_i(x^1)\le 0, \quad g_i(x^2)\le 0,$$

$$g_i(\lambda E(x^1)+(1-\lambda)E(x^2)) \le \lambda b_i(x^1,x^2,\lambda)g_i(x^1)+(1-\lambda b_i(x^1,x^2,\lambda))g_i(x^2) \le 0,$$

$i\in I$. So $\lambda E(x^1)+(1-\lambda)E(x^2)\in M_g$. This shows that M_g is an E-convex set. Thereby, (P_g) is a b-semi-E-convex programming.

Conclusions (ii), (iii) and (iv) are immediate corollaries of Theorems 4.1, 4.2 and 4.3. □

5. Optimality Conditions

In this part, we will discuss the optimality conditions of the b-semi-E-convex programming (P) and (P_g).

Consider the problem (P) firstly. We now develop a necessary and sufficient condition for the global solution. The following Theorem 5.1 is similar to Corollary 2 of Theorem 3.4.3 in [1].

Theorem 5.1. *Let $f : M \to R$ be differentiable and b-semi-E-convex on M, $x^* \in M$. Assume that $E(x^*) = x^*$, and $\bar{b}(y, x^*) = \lim_{\lambda \to 0^+} b(y, x^*, \lambda) > 0$ for all $y \in M$ and $y \neq x^*$. Then x^* is an optimal solution of the b-semi-E-convex programming (P) if and only if*

$$\nabla f(x^*)^T (E(y) - x^*) \geq 0, \quad \forall y \in M. \tag{5.1}$$

Proof. Prove the necessity of Theorem 5.1. Since x^* is an optimal solution of the b-semi-E-convex programming (P), for each $y \in M$, by Taylor expansion, we have

$$\begin{aligned} f(x^*) \leq f(\lambda E(y) + (1-\lambda)E(x^*)) &= f(x^* + \lambda(E(y) - x^*)) \\ &= f(x^*) + \lambda \nabla f(x^*)^T (E(y) - x^*) + o(\lambda), \end{aligned}$$

that is

$$\lambda \nabla f(x^*)^T (E(y) - x^*) + o(\lambda) \geq 0. \tag{5.2}$$

Dividing the inequality (5.2) by λ and taking $\lambda \to 0^+$, we have $\nabla f(x^*)^T (E(y) - x^*) \geq 0$.

Prove the sufficiency of Theorem 5.1. Since f is b-semi-E-convex at point x^*, for any $y \in M$ and $y \neq x^*$, we get

$$\begin{aligned} f(\lambda E(y) + (1-\lambda)E(x^*)) &= f(x^* + \lambda(E(y) - x^*)) \\ &\leq \lambda b(y, x^*, \lambda) f(y) + (1 - \lambda b(y, x^*, \lambda)) f(x^*). \end{aligned}$$

Again,

$$f(x^* + \lambda(E(y) - x^*)) = f(x^*) + \lambda \nabla f(x^*)^T (E(y) - x^*) + o(\lambda).$$

So

$$\lambda \nabla f(x^*)^T (E(y) - x^*) + o(\lambda) \leq \lambda b(y, x^*, \lambda)(f(y) - f(x^*)). \tag{5.3}$$

Dividing the inequality (5.3) by λ and taking $\lambda \to 0^+$, we have $\nabla f(x^*)^T(E(y)-x^*) \leq \bar{b}(y,x^*)(f(y)-f(x^*))$. In view of $\nabla f(x^*)^T(E(y)-x^*) \geq 0$, and $\bar{b}(y,x^*) > 0$ for $y \neq x^*$, we obtain

$$f(y) \geq f(x^*), \quad \forall\, y \in M,\ y \neq x^*,$$

that is, x^* is an optimal solution of the b-semi-E-convex programming (P). □

Now let us take into account the problem (P_g). For convenience of discussion, we assume always that M_g is a nonempty E-convex set in R^n with respect to map $E : M_g \to R^n$.

Theorem 5.2. (Karush-Kuhn-Tucker Sufficient Conditions) *Suppose that x^* is a KKT point of (P_g). For all $x \in M_g$, $x \neq x^*$, assume that function $f : R^n \to R$ is differentiable and pseudo-$\bar{b}$-semi-E-convex at point x^* and $\bar{b}(x,x^*) > 0$, and functions g_i $(i \in I(x^*) = \{i : g_i(x^*) = 0\}) : R^n \to R$ are differentiable and quasi-b_i-semi-E-convex functions at point x^* with respect to maps $b_i(x,x^*,\lambda) > 0$ for $\lambda > 0$ sufficient small. If $E(x^*) = x^*$, then x^* is an optimal solution of the problem (P_g).*

Proof. For any $x \in M_g$ and $x \neq x^*$, we have $g_i(x) \leq 0 = g_i(x^*)$, $i \in I(x^*)$. By the quasi-b_i-semi-E-convexity of $g_i(\ i \in I(x^*))$ at x^*, from (3.2), it follows that

$$b_i(x,x^*,\lambda)g_i(\lambda E(x) + (1-\lambda)E(x^*)) \leq b_i(x,x^*,\lambda)g_i(x^*), \quad \forall i \in I(x^*).$$

Since $b_i(x,x^*,\lambda) > 0$ and $E(x^*) = x^*$, we get, for $\lambda > 0$ small enough,

$$g_i(\lambda E(x) + (1-\lambda)E(x^*)) = g(x^* + \lambda(E(x) - x^*)) \leq g_i(x^*) = 0, \quad \forall i \in I(x^*). \tag{5.4}$$

Therefore, by the differentiability of $g_i(x)$ and (5.4), one gets

$$0 \geq g_i(x^* + \lambda(E(x)-x^*)) = g_i(x^*) + \lambda \nabla g_i(x^*)^T(E(x)-x^*) + o(\lambda), \quad \forall i \in I(x^*).$$

Noticing that $g_i(x^*) = 0$ for $i \in I(x^*)$, one further has

$$\lambda \nabla g_i(x^*)^T(E(x)-x^*) + o(\lambda) \leq 0. \tag{5.5}$$

Dividing the inequality (5.5) by λ and taking $\lambda \to 0^+$, we have

$$\nabla g_i(x^*)^T(E(x)-x^*) \leq 0, \quad \forall i \in I(x^*). \tag{5.6}$$

Thus, using the KKT conditions and multipliers $u_i \geq 0$, one has

$$\begin{aligned} \nabla f(E(x^*))^T(E(x) - E(x^*)) &= \nabla f(x^*)^T(E(x) - x^*) \\ &= -\sum_{i \in I(x^*)} u_i \nabla g_i(x^*)^T(E(x) - x^*) \geq 0. \end{aligned}$$

Hence, from the pseudo-$\bar{b}$-semi-E-convexity of f at x^* and $\bar{b}(x, x^*) > 0, \forall x \in M_g,\ x \neq x^*$, we can conclude $f(x) \geq f(x^*)$, i.e. x^* is an optimal solution of (P_g). □

Corollary 5.1. *Suppose that x^* is a KKT point of (P_g). For $x \in M_g,\ x \neq x^*$, assume that function $f : R^n \to R$ is differentiable and b-semi-E-convex at point x^*, $\bar{b}(x, x^*) = \lim_{\lambda \to 0^+} b(x, x^*, \lambda) > 0$, and functions g_i $(i \in I(x^*))$ are differentiable and b_i-semi-E-convex functions at point x^* with respect to maps $b_i(x, x^*, \lambda) > 0$ for $\lambda > 0$ sufficient small. If $E(x^*) = x^*$, then x^* is an optimal solution of the problem (P_g).*

Proof. Since $f(E(x^*)) = f(x^*)$, we know, from Proposition 3.6, that f is pseudo-$\bar{b}$-semi-E-convex at point x^*. Again, from Proposition 3.5, one knows that functions g_i $(i \in I(x^*))$ are all quasi-b_i-semi-E-convex at point x^*. Therefore, Corollary 5.1 follows from Theorems 5.2. □

6. Duality Theorems

Duality is considered as one of the most important contents in linear programming and nonlinear programming theory (see [3], [1], [8]). In this section, we will study the Wolfe duality theorems of (P_g) under the b-semi-E-convexity. If functions f and g_i $(i \in I)$ are all differentiable on R^n, then the Wolfe dual problem of (P_g) can be expressed as (see [1], [8])

$$(D_g) \qquad \begin{aligned} \max \quad & f(z) + u^T g(z) \\ \text{s.t.} \quad & \nabla f(z) + \nabla g(z) u = 0, \\ & u \geq 0, \end{aligned}$$

where $u = (u_1, u_2, \ldots, u_m)^T$, $g(z) = (\, g_1(z),\ g_2(z),\ \ldots,\ g_m(z))^T$ and $\nabla g(z) = (\, \nabla g_1(z),\ \ \nabla g_2(z),\ \ldots,\ \nabla g_m(z))$. For convenience, we denote the feasible set of (D_g) by $\overline{M}_g = \{(z, u) : \nabla f(z) + \nabla g(z) u = 0,\ u \geq 0\}$.

Theorem 6.1. (Weak Duality Theorem) *Suppose that $f,\ g_i$ $(i \in I) : R^n \to R$ are all differentiable and b_i-semi-E-convex on R^n with respect to maps b_0 and b_i $(i \in I)$. For each $x \in M_g$ and $(z, u) \in \overline{M}_g$, assume that $E(z) = z$, $\bar{b}_i \triangleq \bar{b}_i(x, z) = \lim_{\lambda \to 0^+} b_i(x, z, \lambda)$, $i = 0$ and $i \in I$, and $\bar{b}_0 > 0$, $\dot{b} \triangleq \dot{b}(x, z) \triangleq$*

$\max\ \{-\infty,\ \overline{b}_i,\ i \in I_1(z)\ \} \leq \overline{b}_0 \leq \ddot{b} \triangleq \ddot{b}(x,z) \triangleq \min\ \{+\infty,\ \overline{b}_i,\ i \in I_2(z)\ \}$, *where* $I_1(z) = \{i \in I :\ g_i(z) < 0\}$, $I_2(z) = \{i \in I :\ g_i(z) > 0\}$. *Then* $f(x) \geq f(z) + u^T g(z)$ *holds for each feasible point* x *of* (P_g) *and each feasible point* (z,u) *of* (D_g), *i.e.,*

$$f(x) \geq f(z) + u^T g(z), \quad \forall x \in M_g, \quad \forall (z,u) \in \overline{M}_g. \tag{6.1}$$

Proof. Since $f(x)$ and $g_i(x)$ $(i \in I)$ are b_i-semi-E-convex functions on R^n with respect to maps b_0 and b_i, $E(z) = z$, for $\lambda > 0$ sufficient small, we have

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)E(z)) &= f(\lambda E(x) + (1-\lambda)z) \\ &\leq \lambda b_0 f(x) + (1-\lambda b_0) f(z). \end{aligned}$$

So, combining Taylor expansion, one has

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)z) &= f(z) + \lambda \nabla f(z)^T (E(x) - z) + o(\lambda) \\ &\leq \lambda b_0 f(x) + (1-\lambda b_0) f(z), \end{aligned}$$

that is

$$\lambda \nabla f(z)^T (E(x) - z) + o(\lambda) \leq \lambda b_0 (f(x) - f(z)). \tag{6.2}$$

Dividing the inequality (6.2) by λ and taking $\lambda \to 0^+$, one has

$$\nabla f(z)^T (E(x) - z) \leq \bar{b}_0 (f(x) - f(z)). \tag{6.3}$$

Similarly, we have

$$\nabla g(z)^T (E(x) - z) \leq A\ (g(x) - g(z)), \tag{6.4}$$

with $A = \operatorname{diag}\ (\ \bar{b}_i, \ \ i \in I)$. Also, by the dual feasibility, we have

$$\nabla f(z)^T (E(x) - z) = -(\nabla g(z) u)^T (E(x) - z) = -u^T \nabla g(z)^T (E(x) - z). \tag{6.5}$$

Taking into account $u \geq 0$, $g(x) \leq 0$, from (6.3), (6.4) and (6.5), we get

$$\begin{aligned} \bar{b}_0 (f(x) - f(z)) &\geq \nabla f(z)^T (E(x) - z) = -u^T \nabla g(z)^T (E(x) - z) \\ &\geq u^T A (g(z) - g(x)) = \sum_{i=1}^{m} u_i \bar{b}_i (g_i(z) - g_i(x)) \\ &\geq \sum_{i=1}^{m} u_i \bar{b}_i g_i(z) = \sum_{i \in I_1(z)} u_i \bar{b}_i g_i(z) + \sum_{i \in I_2(z)} u_i \bar{b}_i g_i(z), \end{aligned}$$

that is to say

$$\bar{b}_0(f(x) - f(z)) \geq \sum_{i \in I_1(z)} u_i \bar{b}_i g_i(z) + \sum_{i \in I_2(z)} u_i \bar{b}_i g_i(z). \tag{6.6}$$

In view of $\dot{b} \leq \bar{b}_0 \leq \ddot{b}$, and $\bar{b}_0 > 0$, dividing the inequality (6.6) by $\bar{b}_0$, we get

$$\begin{aligned} f(x) - f(z) \geq \sum_{i \in I_1(z)} \frac{\bar{b}_i}{\bar{b}_0} u_i g_i(z) + \sum_{i \in I_2(z)} \frac{\bar{b}_i}{\bar{b}_0} u_i g_i(z) \\ \geq \sum_{i \in I_1(z)} u_i g_i(z) + \sum_{i \in I_2(z)} u_i g_i(z) = u^T g(z). \end{aligned}$$

That is $f(x) \geq f(z) + u^T g(z)$, and the proof is finished. □

We obtain two corollaries as follows from Theorem 6.1, and their proofs can be completed easily.

Corollary 6.1. *Suppose that the conditions of Theorem 6.1 are satisfied. Then*

(i) $\inf \{ f(x) : x \in M_g \} \geq \sup \{ f(z) + u^T g(z) : (z, u) \in \overline{M}_g \}$;

(ii) if there exist a feasible point $\bar{x}$ of (P_g) and a feasible point $(\bar{z}, \bar{u})$ of (D_g) such that $f(\bar{x}) = f(\bar{z}) + \bar{u}^T g(\bar{z})$, then $\bar{x}$ and $(\bar{z}, \bar{u})$ solve the problem (P_g) and the problem (D_g), respectively;

(iii) if the optimal value of (P_g) equals $-\infty$, then (D_g) has no feasible solution;

(iv) if the optimal value of (D_g) is $+\infty$, then (P_g) has no feasible solution.

Corollary 6.2. *Suppose that f, g_i $(i \in I) : R^n \to R$ are all differentiable and b-semi-E-convex on R^n with respect to a same map b. If $E(z) = z$, and $\lim_{\lambda \to 0^+} b(x, z, \lambda) > 0$, for all $x \in M_g$, $(z, u) \in \overline{M}_g$, then $f(x) \geq f(z) + u^T g(z)$ holds for each feasible point x of (P_g) and each feasible point (z, u) of (D_g), i.e.,*

$$f(x) \geq f(z) + u^T g(z), \quad \forall x \in M_g, \quad \forall (z, u) \in \overline{M}_g.$$

Theorem 6.2 below, referred Strong Duality Theorem, shows that under suitable assumptions, the optimal objective function values of the primal and dual problems are equal.

Theorem 6.2. (Strong Duality Theorem) *Suppose that f, g_i $(i \in I)$: $R^n \to R$ are all differentiable and b_i-semi-E-convex on R^n with respect to maps*

b_0 and b_i $(i \in I)$, and assume that (x^*, u^*) is a KKT pair of (P_g), $E(x^*) = x^*$ and $\bar{b}_0(x, x^*) = \lim_{\lambda \to 0^+} b_0(x, x^*, \lambda) > 0$, $\forall x \in R^n, b_i = b_i(y, x^*, \lambda) > 0$, $i \in I(x^*), \forall\, y \in M_g$, $y \neq x^*$, for $\lambda > 0$ sufficient small (so x^* is an optimal solution of (P_g) from Corollary 5.1). For each $(z, u) \in \overline{M}_g$, if $E(z) = z$, $\bar{b}_0 \triangleq \bar{b}_0(x^*, z) > 0$, and $\bar{b}_i \triangleq \bar{b}_i(x^*, z) = \lim_{\lambda \to 0^+} b_i(x^*, z, \lambda)$, $i \in I$, $\dot{b} \triangleq \dot{b}(x^*, z) \triangleq \max\{-\infty,\ \bar{b}_i,\ i \in I_1(z)\} \leq \bar{b}_0 \leq \ddot{b} \triangleq \dot{b}(x^*, z) \triangleq \min\{+\infty,\ \bar{b}_i,\ i \in I_2(z)\}$, where $I_1(z) = \{i \in I :\ g_i(z) < 0\}$, $I_2(z) = \{i \in I :\ g_i(z) > 0\}$, then (x^*, u^*) is an optimal solution of (D_g) and the optimal values of (P_g) and (D_g) are equal.

Proof. Let $(z, u) \in \overline{M}_g$ and $E(z) = z$. Noting that $f(x)$ is b_0-semi-E-convex on R^n and $E(x^*) = x^*$, for $\lambda > 0$ sufficient small, we have

$$\begin{aligned} f(\lambda E(x^*) + (1-\lambda)E(z)) &= f(\lambda x^* + (1-\lambda)z) = f(z + \lambda(x^* - z)) \\ &\leq \lambda b_0(x^*, z, \lambda) f(x^*) + (1 - \lambda b_0(x^*, z, \lambda)) f(z). \end{aligned}$$

Since $f(x)$ is differentiable, by Taylor expansion, combining the inequality above, one also has

$$\begin{aligned} f(z + \lambda(x^* - z)) = f(z) + \lambda \nabla f(z)^T (x^* - z) + o(\lambda) \leq \lambda b_0(x^*, z, \lambda) f(x^*) \\ + (1 - \lambda b_0(x^*, z, \lambda)) f(z) = f(z) + \lambda b_0(x^*, z, \lambda)(f(x^*) - f(z)), \end{aligned}$$

that is

$$\lambda \nabla f(z)^T (x^* - z) + o(\lambda) \leq \lambda b_0(x^*, z, \lambda)(f(x^*) - f(z)). \tag{6.7}$$

Dividing the inequality (6.7) by λ and taking $\lambda \to 0^+$, we have

$$\nabla f(z)^T (x^* - z) \leq \bar{b}_0(x^*, z)(f(x^*) - f(z)).$$

Similarly, we have $\nabla g(z)^T (x^* - z) \leq A\,(g(x^*) - g(z))$, where $A = \mathrm{diag}(\bar{b}_i(x^*, z), i \in I)$.

On the other hand, since (x^*, u^*) is a KKT pair of (P_g), one has $\nabla f(x^*) + \nabla g(x^*) u^* = 0$, $u^* \geq 0$, this implies (x^*, u^*) is a feasible solution of (D_g). Therefore, taking into account $u \geq 0$, $g(x^*) \leq 0$, $(u^*)^T g(x^*) = 0$, and $\dot{b} \leq \bar{b}_0 \leq \ddot{b}$, $\forall x \in M_g$, $\forall (z, u) \in \overline{M}_g$, we have

$$\begin{aligned} &\bar{b}_0(f(x^*) + (u^*)^T g(x^*) - f(z) - u^T g(z)) \\ &\quad = \bar{b}_0(f(x^*) - f(z) - u^T g(z)) \geq \nabla f(z)^T (x^* - z) - \bar{b}_0 u^T g(z) \\ &\quad = -u^T \nabla g(z)^T (x^* - z) - \bar{b}_0 u^T g(z) \geq u^T A(-g(x^*) + g(z)) - \bar{b}_0 u^T g(z) \\ &\quad = -u^T A g(x^*) + \sum_{i \in I_1(z)} u_i(\bar{b}_i - \bar{b}_0) g_i(z) + \sum_{i \in I_2(z)} u_i(\bar{b}_i - \bar{b}_0) g_i(z) \end{aligned}$$

$$\geq -u^T Ag(x^*) \geq 0.$$

Thus, in view of $\bar{b}_0 > 0$, one has

$$f(x^*) + (u^*)^T g(x^*) - f(z) - u^T g(z) \geq 0, \quad \forall (z,u) \in \overline{M}_g.$$

This shows that (x^*, u^*) is an optimal solution of (D_g). Noting that $(u^*)^T g(x^*) = 0$, one knows the optimal values of (P_g) and (D_g) are equal. □

As a special case of Theorem 6.2, we obtain the following corollary.

Corollary 6.3. *Suppose that $f,\ g_i\ (i \in I) : R^n \to R$ are all differentiable and b-semi-E-convex on R^n with respect to a same map b, and assume that (x^*, u^*) is a KKT pair of (P_g), $E(x^*) = x^*$, $\bar{b}(y, x^*) = \lim_{\lambda \to 0^+} b(y, x^*, \lambda) > 0$, $\forall y \in R^n$ (so x^* is an optimal solution of (P_g) from Corollary 5.1). If $E(z) = z$, $\forall (z,u) \in \overline{M}_g$, then (x^*, u^*) is an optimal solution of (D_g) and the optimal values of (P_g) and (D_g) are equal.*

Furthermore, we present a converse duality theorem as follows.

Theorem 6.3. (Converse Duality Theorem) *Suppose that $f,\ g_i\ (i \in I) : R^n \to R$ are all differentiable and b_i-semi-E-convex on R^n with respect to maps b_0 and b_i $(i \in I)$. Let $\bar{x}$ and (z^*, u^*) be feasible solutions to (P_g) and (D_g), respectively. Assume that $f(\bar{x}) = f(z^*) + (u^*)^T g(z^*)$ and $E(z^*) = z^*$, for each $x \in M_g$, $\overline{b}_i \triangleq \overline{b}_i(x, z^*) = \lim_{\lambda \to 0^+} b_i(x, z^*, \lambda)$, $i = 0$ and $i \in I$, and $\overline{b}_0 > 0$, $\dot{b} \triangleq \dot{b}(x, z^*) \triangleq \max\{-\infty,\ \overline{b}_i,\ i \in I_1(z^*)\} \leq \overline{b}_0 \leq \ddot{b} \triangleq \ddot{b}(x, z^*) \triangleq \min\{+\infty,\ \overline{b}_i,\ i \in I_2(z^*)\}$, where $I_1(z^*) = \{i \in I :\ g_i(z^*) < 0\}$, $I_2(z^*) = \{i \in I :\ g_i(z^*) > 0\}$. Then $\bar{x}$ is an optimal solution of (P_g).*

Proof. From the given conditions, combining Theorem 6.1, it follows that

$$f(x) \geq f(z^*) + (u^*)^T g(z^*) = f(\bar{x}),\ \forall x \in M_g.$$

This shows that $\bar{x}$ is an optimal solution of (P_g). □

As a special case of Theorem 6.3, we obtain the following corollary.

Corollary 6.4. *Suppose that $f,\ g_i\ (i \in I) : R^n \to R$ are all differentiable and b-semi-E-convex on R^n with respect to a same map b. Let $\bar{x}$ and (z^*, u^*) be feasible solutions to (P_g) and (D_g), respectively. If $f(\bar{x}) = f(z^*) + (u^*)^T g(z^*)$, $E(z^*) = z^*$ and $\lim_{\lambda \to 0^+} b(x, z^*, \lambda) > 0$, $\forall x \in M_g$. Then $\bar{x}$ is an optimal solution of (P_g).*

7. Acknowledgements

This project is supported by the National Natural Science Foundation (No. 10261001) and Guangxi Science Foundation (Nos. 0236001, 0249003) of P.R. China.

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