International Journal of Pure and Applied Mathematics
Volume 22 No. 2 2005, 155-162

HADAMARD PRODUCTS OF CERTAIN p-VALENTLY ANALYTIC FUNCTIONS

Maslina Darus
School of Mathematical Sciences
Faculty of Science and Technology
University Kebangsaan Malaysia
Bangi, Selangor Darul Ehsan, 43600, MALAYSIA
e-mail: maslina@pkrisc.cc.ukm.my

Abstract: In this paper, we introduce and study a class of p-valently functions with negative coefficients defined by using Hadamard products in the sense of Juneja [5]. Some basic properties which include coefficient bounds, growth and distortion, and closure theorem are given. Further, the radii of starlikeness and convexity is also given.

AMS Subject Classification: 30C45
Key Words: convex, starlike, p-valently functions, Hadamard product, integral operator, fractional calculus

1. Introduction and Preliminaries

Denote by $A(p)$ the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}\,, \tag{1.1}$$

which are analytic in the open disc $U = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. Some properties of some subclasses of $A(p)$ were studied by Aouf et al [1]. Denote by $S^*(p,\alpha)$ the class of starlike functions $f \in A(p)$ of order $\alpha (0 \leq \alpha < p)$ satifying

Received: May 29, 2005

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U\,,$$

and let $C(p,\alpha)$ be the class of convex functions $f \in A(p)$ of order $\alpha(0 \le \alpha < p)$ such that $zf' \in S^*(p,\alpha)$.

A function $f \in A(1)$ is said to be in the class of β-uniformly convex functions of order α, denoted by $\beta - UCV(\alpha)$ [8, 9] if

$$\mathrm{Re}\ \left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \ge \beta \left|\frac{zf''(z)}{f'(z)} - 1\right|, \tag{1.2}$$

and is said to be in a corresponding subclass of $\beta - UCV(\alpha)$ denote by $\beta - S_p(\alpha)$ if

$$\mathrm{Re}\ \left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \ge \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{1.3}$$

where $-1 \le \alpha \le 1$ and $z \in U$.

The class of uniformly convex and uniformly starlike functions has been extensively studied by Goodman [3, 4], Ma and Minda [7]. In fact the class of uniformly β-starlike funtions was introduced by Kanas and Wisniowski [6], and for which it can be generalised to $\beta - S_p(\alpha)$, the class of uniformly β-starlike functions of order α.

If f of the form (1.1) and $g(z) = z^p + \sum\limits_{k=1}^{\infty} b_{p+k}z^{p+k}$ are two functions in $A(p)$, then the Hadamard product (or convolution) of f and g is denoted by $f * g$ and is given by

$$(f*g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}b_{p+k}z^{p+k}. \tag{1.4}$$

Ruscheweyh [10] using the convolution techniques, introduced and studied an important subclass of $A(1)$, the class of prestarlike functions of order α, which denoted by $\mathcal{R}(\alpha)$. Thus $f \in A(1)$ is said to be prestarlike function of order $\alpha(0 \le \alpha < 1)$ if $f * S_\alpha \in S^*(\alpha)$, where $S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} c_n(\alpha)z^n$ and $c_n(\alpha) = \frac{\Pi_{j=2}^{n}(j-2\alpha)}{(n-1)!}$ $\quad (n \in \mathbf{N}\,\{1\} \quad \mathbf{N} := \{1,2,3,\ldots\})$. We note that $\mathcal{R}(0) = C(0)$ and $\mathcal{R}(\frac{1}{2}) = S^*(\frac{1}{2})$. Juneja et al [5] define the family $\mathcal{D}(\Phi,\Psi;\alpha)$ consisting of functions $f \in A$ so that

$$\mathrm{Re}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}\right) > \alpha, \quad z \in U\,,$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in U such that $f(z) * \Psi(z) \ne 0$, $\Upsilon_n \ge 0$, $\gamma_n \ge 0$ and $\Upsilon_n > \gamma_n$ $(n \ge 2)$.

We let $\mathcal{D}(\Phi, \Psi; \eta, \beta, p)$ denote the set of all functions in $A(p)$ for which

$$\operatorname{Re}\left(1+\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right)>\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right|, \qquad (1.5)$$

where η is positive real number and $\beta \geq 0$. In the case of $p = 1$, this similar class of function was introduced in [2]. Thus, the work here is more or less the generalisation of the work in [2].

For suitable choices of Φ, Ψ, and having $\eta = p - \alpha$, we easily obtain the various subclasses of $A(p)$. For example $\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; p-\alpha, 0) = S^*(p, \alpha)$, $\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; p-\alpha, 0) = C(p, \alpha)$, $\mathcal{D}(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; p-\alpha, 0) = \mathcal{R}(p, \alpha)$, $\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; p-\alpha, \beta) = \beta - S_p(p, \alpha)$, and $\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; p-\alpha, \beta) = \beta - UCV(p, \alpha)$. And having in mind that when $p = 1$ we obtain

$$\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 1-\alpha, 0) = S^*(\alpha),$$

$$\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 1-\alpha, 0) = C(\alpha),$$

$$\mathcal{D}(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; 1-\alpha, 0) = \mathcal{R}(\alpha),$$

$$\mathcal{D}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 1-\alpha, \beta) = \beta - S_p(\alpha),$$

and

$$\mathcal{D}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 1-\alpha, \beta) = \beta - UCV(\alpha).$$

Also denote by $T(p)$ [11] the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}. \qquad (1.6)$$

Now let us write $\mathcal{D}_T(\Phi, \Psi; \eta, \beta, p) = \mathcal{D}(\Phi, \Psi; \eta, \beta, p) \cap T$.

2. Characterization Property

We now investigate the characterization property for the function f to be in $\mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$. First of all, we get the coefficient bounds which shall be used throughout the paper.

Theorem 2.1. (Coefficient Bounds) *A function f defined by (1.6) is in the class $\mathcal{D}_T(\Phi,\Psi;\eta,\beta,p)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{[(1+\beta)\Upsilon_{p+k}-(p(1+\beta)-\eta)\gamma_{p+k}]}{\eta-(1+\beta)(p-1)}|a_{p+k}|\leq 1\,, \tag{2.7}$$

where η is positive real number, $\beta\geq 0$, $\Upsilon_{p+k}\geq 0$, $\gamma_{p+k}\geq 0$ and $\Upsilon_{p+k}>\gamma_{p+k}$.

Proof. It suffices to show that

$$\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right|\leq \mathrm{Re}\,\left\{1+\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right\},$$

and we have

$$\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right|\leq \mathrm{Re}\,\left\{\left\{1+\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right\}-1\right\}+1.$$

That is

$$\beta\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right|-\mathrm{Re}\,\left\{\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right\}$$
$$\leq(\beta+1)\left|\frac{1}{\eta}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-p\right)\right|$$
$$\leq\frac{(1+\beta)(p-1)+\sum_{n=2}^{\infty}(1+\beta)(\Upsilon_{p+k}-p\gamma_{p+k})|a_{p+k}|}{\eta-\sum_{k=1}^{\infty}\eta\gamma_{p+k}|a_{p+k}|}.$$

The above expression is bounded by 1 and hence the assertion of the result.

Now we need to show that $f\in\mathcal{D}_T(\Phi,\Psi;\eta,\beta,p)$ satisfies the coefficient inequality. If $\mathcal{D}_T(\Phi,\Psi;\eta,\beta,p)$ and z is real then (1.5) yields

$$1+\frac{1}{\eta}\left(\frac{1-\sum_{k=1}^{\infty}\Upsilon_{p+k}a_{p+k}z^k}{1-\sum_{k=1}^{\infty}\gamma_{p+k}a_{p+k}z^k}-p\right)\geq\frac{\beta(p-1)+\sum_{k=1}^{\infty}\beta(\Upsilon_{p+k}-p\gamma_{p+k})a_{p+k}z^k}{\eta-\sum_{k=1}^{\infty}\eta\gamma_{p+k}a_{p+k}z^k}.$$

Letting $z\to 1$ along the real axis leads to the desired inequality

$$\sum_{k=1}^{\infty}[(1+\beta)\Upsilon_{p+k}-(p(1+\beta)-\eta)\gamma_{p+k}]a_{p+k}\leq\eta-(1+\beta)(p-1). \qquad \square$$

Corollary 2.1. *Let a function f defined by (1.6) belong to the class $\mathcal{D}_T(\Phi,\Psi;\eta,\beta,p)$. Then*

$$a_{p+k} \leq \frac{\eta-(1+\beta)(p-1)}{[(1+\beta)\Upsilon_{p+k}-(p(1+\beta)-\eta)\gamma_{p+k}]}, \quad k \geq 1.$$

For $\beta=0$ and $\eta=p-\alpha$, we obtain the following corollary.

Corollary 2.2. *Let a function f defined by (1.6) belong to the class $\mathcal{D}_T(\Phi,\Psi;p-\alpha,0)$. Then*

$$\sum_{k=1}^{\infty}\frac{[\Upsilon_{p+k}-\alpha\gamma_{p+k}]}{1-\alpha}|a_{p+k}| \leq 1. \tag{2.8}$$

For $\beta=0$ and $\eta=1-\alpha$ we have result obtained by Juneja [5].

Corollary 2.3. (see [5]) *Let a function f defined by (1.6) belong to the class $\mathcal{D}_T(\Phi,\Psi;1-\alpha,0)$. Then*

$$\sum_{k=1}^{\infty}\frac{[\Upsilon_{1+k}-\alpha\gamma_{1+k}]}{1-\alpha}|a_{1+k}| \leq 1. \tag{2.9}$$

Next we consider the growth and distortion theorem for the class $\mathcal{D}_T(\Phi,\Psi;\eta,\beta,p)$. We shall omit the proof as the techniques are tedious and standard.

Theorem 2.2. *Let the function f defined by (1.6) be in the class $\mathcal{D}_T(\Phi,\Psi;\eta,\beta,p)$. Then*

$$r^p - r^{p+1}\frac{\eta-(1+\beta)(p-1)}{[(1+\beta)\Upsilon_{p+1}-(p(1+\beta)-\eta)\gamma_{p+1}]} \leq |f(z)| \leq r + r^{p+1}\frac{\eta-(1+\beta)(p-1)}{[(1+\beta)\Upsilon_{p+1}-(p(1+\beta)-\eta)\gamma_{p+1}]} \quad (|z|=r) \tag{2.10}$$

and

$$pr^{p-1} - r^p\frac{(p+1)(\eta-(p-1)(1+\beta))}{[(1+\beta)\Upsilon_{p+1}-(p(1+\beta)-\eta)\gamma_{p+1}]} \leq |f'(z)| \leq pr^{p-1} + r^p\frac{(p+1)(\eta-(p-1)(1+\beta))}{[(1+\beta)\Upsilon_{p+1}-(p(1+\beta)-\eta)\gamma_{p+1}]} \quad (|z|=r). \tag{2.11}$$

The bounds (2.10) and (2.11) are attained for functions given by

$$f(z) = z^p - z^{p+k}\frac{\eta-(1+\beta)(p-1)}{[(1+\beta)\Upsilon_{p+k}-(p(1+\beta)-\eta)\gamma_{p+k}]} \quad (k \geq 1). \tag{2.12}$$

Theorem 2.3. *Let a function f be defined by (1.6) and*

$$g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \tag{2.13}$$

be in the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$. Then the function h defined by

$$h(z) = (1-\lambda) f(z) + \lambda g(z) = z^p - \sum_{k=1}^{\infty} q_{p+k} z^{p+k}, \tag{2.14}$$

where $q_{p+k} = (1-\lambda) a_{p+k} + \lambda b_{p+k}$, $0 \leq \lambda \leq 1$ is also in the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$.

Proof. The result follows easily from (2.7) and (2.14). □

We prove the following theorem by defining functions $f_j(z) (j = 1, 2, \ldots m)$ of the form

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \text{ for } a_{p+k,j} \geq 0, \ z \in U \tag{2.15}$$

Theorem 2.4. (Closure Theorem) *Let the functions $f_j(z) (j = 1, 2 \ldots m)$ defined by (2.15) be in the classes $\mathcal{D}_T(\Phi, \Psi; \eta_j, \beta, p)) (j = 1, 2, \ldots m)$, respectively. Then the function $h(z)$ defined by $h(z) = z^p - \frac{1}{m} \sum_{k=1}^{\infty} (\sum_{j=1}^{m} a_{p+k,j})$ z^{p+k} is in the class $\mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$, where*

$$\eta = \min_{1 \leq j \leq m} \{\eta_j\} \quad \text{with} \quad 0 \leq \eta_j < 1. \tag{2.16}$$

Proof. Since $f_j \in \mathcal{D}_T(\Phi, \Psi; \eta_j, \beta, p) (j = 2, \ldots m)$ by applying Theorem 2.1, we observe that

$$\begin{aligned}
&\sum_{k=1}^{\infty} [(1+\beta) \Upsilon_{p+k} - (p(1+\beta) - \eta) \gamma_{p+k}] (\frac{1}{m} \sum_{j=1}^{m} a_{p+k,j}) \\
&= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} [(1+\beta) \Upsilon_{p+k} - (p(1+\beta) - \eta) \gamma_{p+k}] a_{p+k,j} \right) \\
&\leq \frac{1}{m} \sum_{j=1}^{m} (\eta_j - (1+\beta)(p-1)) \leq \eta - (1+\beta)(p-1),
\end{aligned}$$

which in view of Theorem 2.1, again implies that $h \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$ and the proof is complete. □

3. Radii of Starlikeness and Convexity of $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$

We first provide the radii of starlikeness for functions $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$.

Theorem 3.1. *Let $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta)$. Then f is starlike of order $0 \leq \tau < p$ in $|z| < R_1$, where*

$$R_1 = \inf_{k \geq 1} \left[\frac{(p-\tau)[(1+\beta)\Upsilon_{p+k} - (p(1+\beta)-\eta)\gamma_{p+k}]}{(p+k-\tau)[\eta - (1+\beta)(p-1)]} \right]^{\frac{1}{p+k-1}}.$$

Proof. It is sufficient to prove

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \tau. \tag{3.1}$$

For the left hand side of (3.1) we have

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{\sum_{k=1}^{\infty} (-k) a_{p+k} z^{p+k-1}}{1 - \sum_{k=2}^{\infty} a_{p+k} z^{p+k-1}} \right| \leq \frac{\sum_{k=1}^{\infty} k a_{p+k} |z|^{p+k-1}}{1 - \sum_{k=1}^{\infty} a_{p+k} |z|^{p+k-1}}.$$

This last expression is less than $(p - \tau)$ since

$$|z|^{p+k-1} < \frac{(p-\tau)[(1+\beta)\Upsilon_{p+k} - (p(1+\beta)-\eta)\gamma_{p+k}]}{(p+k-\tau)[\eta - (1+\beta)(p-1)]}.$$

Therefore the proof is completed. □

Using the fact that f is convex if and only if zf' is starlike, we obtain the following theorem.

Theorem 3.2. *Let $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$. Then f is convex of order $0 \leq \tau < p$ in $|z| < R_2$, where*

$$R_2 = \inf_{k \geq 1} \left[\frac{(p-\tau)[(1+\beta)\Upsilon_{p+k} - (p(1+\beta)-\eta)\gamma_{p+k}]}{(p+k)(p+k-\tau)[\eta - (1+\beta)(p-1)]} \right]^{\frac{1}{p+k-1}}.$$

We omit the proof as it is easily derived.

Finally we obtain the following result.

Theorem 3.3. *Let $f \in \mathcal{D}_T(\Phi, \Psi; \eta, \beta, p)$. Then f is close-to-convex of order $0 \leq \tau < p$ in $|z| < R_3$, where*

$$R_3 = \inf_{k \geq 1} \left[\frac{(p-\tau)[(1+\beta)\Upsilon_{p+k} - (p(1+\beta)-\eta)\gamma_{p+k}]}{(p+k)[\eta - (1+\beta)(p-1)]} \right]^{\frac{1}{p+k-1}}.$$

Acknowledgements

The work presented here is supported by IRPA grant 04-02-02-10029 EAR, Malaysia.

References

[1] M.K. Aouf, M. Hossen, H.M. Srivastava, Some families of univalent functions, *Appl. Math. Comput.*, **39** (2000), 39-48.

[2] M. Darus, Hadamard product of certain uniformly analytic functions, Submitted.

[3] A.W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87-92.

[4] A.W. Goodman, On uniformly starlike functions, *J. Math. Anal. and Appl.*, **155** (1991), 364-370.

[5] O.P. Juneja, T.R. Reddy, M.L. Mogra, A convolution approach for analytic functions with negative coefficients, *Soochow J. Math.*, **11** (1985), 69-81.

[6] S. Kanas, A. Wisniowska, Conic domains and starlike functions, *Rev. Roumaine Math. Pures Appl.*, **45** (2000), 647-657.

[7] W. Ma, D. Minda, Uniformly convex functions, *Ann. Polon. Math.*, **57** (1992), 165-175.

[8] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189-196.

[9] F. Rønning, Integral representations for bounded starlike functions, *Annal. Polon. Math.*, **60** (1995), 289-297.

[10] S. Ruscheweyh, Linear operator between classes of prestarlike functions, *Comm. Math. Helv.*, **52** (1977), 497-509.

[11] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109-116.