

ON STRONG COMPARISON PRINCIPLE FOR
SEMICONTINUOUS VISCOSITY SOLUTIONS OF
SOME NONLINEAR ELLIPTIC EQUATIONS

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Abstract: The strong comparison principle for semicontinuous viscosity solutions of some nonlinear elliptic equations are considered. For linear elliptic equations it is well known that the strong comparison principle is equivalent to the strong maximum principle. However, for nonlinear equations the strong maximum principle may not imply the strong comparison principle. We establish a strong comparison principle for some nonlinear elliptic equations including the minimal surface equation.

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1. Introduction

We are concerned with an elliptic equation of the form

$$F(Du(x), D^2u(x)) = 0 \quad \text{in } \Omega, \quad (1.1)$$

where Ω is a domain in \mathbb{R}^n . The function $u : \Omega \rightarrow \mathbb{R}$ is unknown and F is a

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given function. Here Du and D^2u denote, respectively, the gradient of u and the Hessian of u in variables x . The function $F : \mathbb{R}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ is continuous, where \mathbb{S}^n denotes the space of all real $n \times n$ symmetric matrices.

Our goal is to establish the strong comparison principle for viscosity solutions of (1.1). By the strong comparison principle we mean the principle that a subsolution u agrees with a supersolution v in Ω if $u \leq v$ in Ω and $u(x_0) = v(x_0)$ at some point $x_0 \in \Omega$. A typical example of $F = F(p, X)$ we consider here is of the form

$$F(p, X) = -\text{trace} \left\{ \left(I - \frac{p \otimes p}{1 + |p|^2} \right) X \right\},$$

so that (1.1) becomes

$$-\sqrt{1 + |Du|^2} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0 \quad \text{in } \Omega. \quad (1.2)$$

The equation (1.2) is called the (graph) minimal surface equation.

We shall establish the strong comparison principle for some elliptic equations including the graph minimal surface equation. A solution we consider here is a viscosity solution which may not be continuous. The idea of our proof of the strong comparison principle reflects that of the classical strong maximum principle to uniformly linear elliptic equations (cf. [9], [5]). By the strong maximum principle we mean the principle that a subsolution u equals a constant M if $u \leq M$ in Ω and $u(x_0) = M$ at some point $x_0 \in \Omega$. Evidently the strong comparison principle implies the strong maximum principle provided that a constant is a solution. However, as we shall see later (Remark 2.6) the converse may not hold. To prove the strong maximum principle one only need to study the relation of a subsolution and its maximum value. However, to prove the strong comparison principle we have to study the relation between a subsolution and a supersolution of the equation. So we are forced to choose a test function by doubling variables. Since we use an auxiliary function as in the proof of the classical strong maximum principle for linear elliptic equations (cf. [9], [5]), our test function $\Phi(x, y)$ loses symmetry of variables between $x \in \Omega$ and $y \in \Omega$. We need some efforts for estimates of matrices concerned with $D^2\Phi(x, y)$. We also establish the Hopf Boundary Lemma. Its proof is very similar to that of the strong comparison principle.

For general $F(p, X)$ we need some ellipticity. If we remove elliptic condition completely we have a counterexample to the strong comparison principle. For $|\frac{du}{dx}| = 1$ on $(-1, 1)$ there are two solutions and those are not coincide on $(-1, 1)$ (see Remark 2.6). Moreover, we need some Lipschitz condition on p . When we

lose the Lipschitz condition, we have a counterexample. In fact, for $-\Delta u - |Du|^m = 0$ in $B(0, R)$ with $0 < m < 1$ there is a nonconstant solution which attains its maximum at center 0 and the value is zero on the boundary of the ball $B(0, R)$. In this case even the strong maximum principle does not hold.

It is well known that for linear elliptic equations the strong comparison principle is equivalent to the strong maximum principle since linear combinations of solutions are still solutions. The strong maximum principle of classical solutions for linear elliptic equations has been well studied (cf. [9], [5]). There are a few results on the strong maximum principle for weak solution (distribution sense) of quasilinear possibly degenerate equations (see e.g. [11], [8], [5]). For viscosity solutions Kawohl and Kutev [7] prove the strong maximum principle under continuity condition for subsolutions or supersolutions. Later, Bardi and Da Lio [1] improve this result without continuity assumption for solutions and they establish the strong maximum principle for a large class including the graph minimal surface equation and even for degenerate elliptic equations, for example, for the p -Laplacian equation with $p > 1$. For a level set equation of the minimal surface equation a special form of a strong maximum principle for level sets of solutions was established by [4]. Our choice of test function is close to that of [7]. In [1] to prove the strong maximum principle they just consider the relation of a subsolution $u(x)$ and its maximum. So they do not have to consider the test function $\Phi(x, y)$ we use; they need not to estimate the second derivative $D^2\Phi(x, y)$. For viscosity solutions Trudinger [10] proved the strong comparison principle for uniformly elliptic equations with Lipschitz continuity assumptions on subsolutions and supersolutions. He only state results in [10, Remark 3.2] without the proof.

This paper is organized as follows. In Section 2 we recall a notion of viscosity solutions through definition of viscosity subsolutions and supersolutions to elliptic differential equations. Then we list assumptions on $F = F(p, X)$. We give some comments to our assumptions. In Section 3 we establish the strong comparison principle of viscosity solutions. In Section 4 we prove the Hopf boundary lemma for viscosity solutions. In Section 5 we give some examples of equation (1.1). There we verify if our assumptions hold to those examples.

After this work was completed, we were informed of a recent work of Ishii and Yoshimura [6] who proved the strong comparison principle for semicontinuous viscosity solutions of uniformly elliptic equations. Their proof is very similar to ours.

2. Definition of Viscosity Solutions and Assumptions

on $F = F(p, X)$

Let Ω be a domain in \mathbb{R}^n . We consider a elliptic equation of form

$$F(Du(x), D^2u(x)) = 0 \quad \text{in } \Omega. \quad (2.1)$$

Here Du and D^2u denote, respectively, the gradient of u and the Hessian of u in variables x .

Now we recall a definition of viscosity solutions of (2.1). We list the basic assumptions on $F = F(p, X)$.

$$(F1) \quad F : \mathbb{R}^n \times \mathbb{S}^n \rightarrow \mathbb{R} \quad \text{is continuous,}$$

where \mathbb{S}^n denotes the space of all real $n \times n$ symmetric matrices.

We will use the following notations;

$$\text{USC}(\Omega) = \{\text{upper semicontinuous functions } u : \Omega \rightarrow \mathbb{R}\},$$

$$\text{LSC}(\Omega) = \{\text{lower semicontinuous functions } u : \Omega \rightarrow \mathbb{R}\}.$$

Definition 2.1. Let $u : \Omega \rightarrow \mathbb{R}$.

(i) A function $u \in \text{USC}(\Omega)$ is a viscosity subsolution of (2.1), if for all $\varphi \in C^2(\Omega)$ and local maximum points \hat{x} of $u - \varphi$ we have

$$F(D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq 0.$$

(ii) A function $u \in \text{LSC}(\Omega)$ is a viscosity supersolution of (2.1), if for all $\varphi \in C^2(\Omega)$ and local minimum points \hat{x} of $u - \varphi$ we have

$$F(D\varphi(\hat{x}), D^2\varphi(\hat{x})) \geq 0.$$

(iii) If u is a viscosity subsolution and a viscosity supersolution of (2.1), then we call u a viscosity solution of (2.1).

We recall one of equivalent definition of viscosity sub- and supersolutions of (2.1) (cf. [3]).

Definition 2.2. Let $u : \Omega \rightarrow \mathbb{R}$.

(i) A function $u \in \text{USC}(\Omega)$ is a viscosity subsolution of (2.1),

$$F(p, X) \leq 0 \quad \text{for all } (p, X) \in J^{2,+}u(x), \quad x \in \Omega.$$

(ii) A function $u \in \text{LSC}(\Omega)$ is a viscosity supersolution of (2.1),

$$F(p, X) \geq 0 \quad \text{for all } (p, X) \in J^{2,-}u(x), \quad x \in \Omega.$$

Here $J^{2,+}$ denotes the elliptic super 2-jet in Ω , i.e., $J^{2,+}$ is the set of $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$ that satisfies

$$u(y) \leq u(x) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2)$$

as $y \rightarrow x$ in Ω ,

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Similarly, $J^{2,-}$ denotes the elliptic sub 2-jet in Ω , i.e., $J^{2,-}$ is the set of $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$ that satisfies

$$u(y) \geq u(x) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2)$$

as $y \rightarrow x$ in Ω .

Note that $J^{2,+}$ and $J^{2,-}$ have the relation $J^{2,-}u = -J^{2,+}(-u)$.

We next describe of a class of equations for which we shall establish a strong comparison principle. We shall introduce a notion called *coercive*.

Definition 2.3. We say that a function $f : \mathbb{R} \times \mathbb{S}^n \rightarrow \mathbb{R}$ is *coercive* if for each $M > 0$ there exists a function $\beta = \beta_M : [0, \infty) \rightarrow \mathbb{R}$ satisfying:

$$\beta \text{ is continuous on } [0, \infty) \text{ and } \lim_{\sigma \rightarrow +\infty} \beta(\sigma) = +\infty, \tag{i}$$

$$f(p, S) \geq b\beta(N) \tag{ii}$$

for all $S \in \mathbb{S}^n$, $b > 0$, $N > 0$ and $p \in \mathbb{R}^n$ satisfying

$$S \leq bI, \quad {}^t\mu S \mu \leq -bN, \quad |p| \leq M$$

for some $\mu \in S^{n-1}$. Here I denotes the identity matrix, μ is a row vector, ${}^t\mu$ is the transposed vector of μ and S^{n-1} denotes the set of unit vectors in \mathbb{R}^n . The function β is called a *bound* for f .

We shall assume a kind of ellipticity and a Lipschitz continuity of derivative variables p for $F = F(p, X)$.

(F2) There exists a coercive function f such that

$$F(p, X) - F(p, -Y) \geq f(p, X + Y)$$

for all $p \in \mathbb{R}^n$ and for all $X, Y \in \mathbb{S}^n$.

(F3) Let M and K be positive. There exists a positive constant $L_{M,K}$ such that

$$|F(q, X) - F(\tilde{q}, X)| \leq L_{M,K}|q - \tilde{q}|$$

for all $q, \tilde{q} \in \mathbb{R}^n$ satisfying $|q|, |\tilde{q}| \leq M$ and for all $X \in \mathbb{S}^n$ satisfying $\|X\| \leq K$, where $\|X\|$ denotes the operator norm of X as a self-adjoint operator on \mathbb{R}^n .

We shall see that the uniform ellipticity implies (F2). Let us recall a definition of uniformly elliptic equations. Let M be positive. If there exists constant $0 < \lambda_M \leq \Lambda_M$ such that

$$\lambda_M \text{ trace } Y \leq F(p, X - Y) - F(p, X) \leq \Lambda_M \text{ trace } Y \quad (2.2)$$

for all $p \in \mathbb{R}^n$ satisfying $|p| \leq M$, $X, Y \in \mathbb{S}^n$ and $Y \geq 0$, then we call $F = F(p, X)$ is *uniformly elliptic*. It turns out that (F2) is fulfilled if $F = F(p, X)$ is uniformly elliptic (Proposition 2.5). Let λ_j ($1 \leq j \leq n$) be the set of eigenvalues of X including multiplicity. Let e_j be eigenvectors of λ_j . We may assume that $\{e_j\}_{j=1}^n$ is an orthogonal basis of \mathbb{R}^n . Thus we have a spectral decomposition

$$X = \sum_{j=1}^n \lambda_j e_j \otimes e_j.$$

We define the plus part X_+ and minus part X_- by

$$X_+ := \sum_{j=1}^n (\lambda_j)_+ e_j \otimes e_j, \quad X_- := \sum_{j=1}^n (\lambda_j)_- e_j \otimes e_j,$$

where $(\lambda_j)_+ := \max\{0, \lambda_j\}$ and $(\lambda_j)_- := \min\{0, \lambda_j\}$.

Proposition 2.4. *Let F be uniformly elliptic. Then we have*

$$F(p, X) - F(p, -Y) \geq -\Lambda_M \text{ trace } (X + Y)_+ - \lambda_M \text{ trace } (X + Y)_-.$$

Proof. Let

$$Z = \sum_{j=1}^n \lambda_j e_j \otimes e_j$$

be a spectral decomposition of $Z = X + Y$. We may assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$ and $\lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_{\ell+1} < 0$. We calculate

$$\begin{aligned} F(p, X) - F(p, -Y) &= F(p, X) - F(p, X - Z) \\ &= F(p, X) - F(p, X - \sum_{j=1}^n \lambda_j e_j \otimes e_j) = F(p, X) - F(p, X - \lambda_1 e_1 \otimes e_1) \\ &\quad + F(p, X - \lambda_1 e_1 \otimes e_1) - F(p, X - (\lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2)) \end{aligned}$$

$$\begin{aligned}
 & \vdots \\
 & +F(p, X - \sum_{j=1}^{\ell-1} \lambda_j e_j \otimes e_j) - F(p, X - \sum_{j=1}^{\ell} \lambda_j e_j \otimes e_j) \\
 & +F(p, X - \sum_{j=1}^{\ell} \lambda_j e_j \otimes e_j) - F(p, X - \sum_{j=1}^{\ell+1} \lambda_j e_j \otimes e_j) \\
 & \vdots \\
 & +F(p, X - \sum_{j=1}^{n-1} \lambda_j e_j \otimes e_j) - F(p, X - \sum_{j=1}^n \lambda_j e_j \otimes e_j)
 \end{aligned}$$

and apply (2.2) to get

$$\begin{aligned}
 & \geq -\Lambda_M \lambda_1 - \Lambda_M \lambda_2 - \dots - \Lambda_M \lambda_\ell + \lambda_M |\lambda_{\ell+1}| + \dots + \lambda_M |\lambda_n| \\
 & \quad = -\Lambda_M (\lambda_1 + \dots + \lambda_\ell) - \lambda_M (\lambda_{\ell+1} + \dots + \lambda_n) \\
 & \quad = -\Lambda_M \text{trace } Z_+ - \lambda_M \text{trace } Z_-. \quad \square
 \end{aligned}$$

As we prove later (see Section 5) $-\Lambda_M \text{trace } (X+Y)_+ - \lambda_M \text{trace } (X+Y)_-$ is a coercive function for uniformly elliptic equations. Thus by Proposition 2.4 we have the following result.

Proposition 2.5. *Let F be uniformly elliptic. Then F satisfies (F2).*

Remark 2.6. (i) For the strong comparison principle one cannot remove (F2) completely. In fact the strong comparison principle fails for a first order equation $|\frac{du}{dx}| = 1$ on $(-1, 1)$ which does not fulfill (F2). Indeed there are solutions $u_1(x) = x + 1$ and $u_2(x) = -|x| + 1$. We observe that $u_1(x) \geq u_2(x)$ on $(-1, 1)$ and $u_1(x) \equiv u_2(x)$ on $(-1, 0)$. However, $u_1(x) > u_2(x)$ on $(0, 1)$. This means that the strong comparison principle is not fulfilled.

(ii) One would like to weaken the Lipschitz condition of $F(p, X)$ in p . For example, we consider

$$|F(q, X) - F(\tilde{q}, X)| \leq L_{M,K} |q - \tilde{q}|^m$$

for some m ($0 < m < 1$). However, for such F we have a counterexample (cf. [2]). Let $0 < m < 1, R > 0$,

$$F(p, X) = -\text{trace } X - |p|^m, \quad \Omega = B(0, R) \subset \mathbb{R}^n.$$

For this F equation (2.1) becomes

$$-\Delta u - |Du|^m = 0 \quad \text{in } B(0, R). \tag{2.3}$$

In [1] there is a comment to (2.3). For (2.3) the strong minimum principle holds, however the strong maximum principle does not hold. In fact, $u(x) = C(R^k - |x|^k)$ with $k = (2 - m)/(1 - m)$, $C = k^{-1}(n + k - 2)^{1/(m-1)}$ is a non constant solution to (2.3) (cf. [2]). This means for (2.3) that the strong comparison principle does not hold. So we cannot remove the Lipschitz continuity assumption completely. If we would like to weaken the assumption (F3), we have to consider another way.

Remark 2.7. A typical example is the minimal surface equation

$$-\sqrt{1 + |Du|^2} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0 \quad \text{in } \Omega. \quad (2.4)$$

For this equation $F = F(p, X)$ is given by

$$F(p, X) = -\operatorname{trace} \left\{ \left(I - \frac{p \otimes p}{1 + |p|^2} \right) X \right\}. \quad (2.5)$$

This $F = F(p, X)$ is uniformly elliptic. Indeed, for (2.5) elliptic constants are taken by $\lambda_M = 1/(1 + M^2)$, $\Lambda_M = 1$. An extended equation of (2.4) is the following.

$$-\operatorname{trace} \left\{ A(Du) \left(I - \frac{Du \otimes Du}{1 + |Du|^2} \right) D^2u \left(I - \frac{Du \otimes Du}{1 + |Du|^2} \right) \right\} = 0 \quad \text{in } \Omega, \quad (2.6)$$

where $A(p) \in \mathbb{S}^n$ satisfies $A(p) \geq 0$ for all $p \in \mathbb{R}^n$. We shall assume that for each $M > 0$ there exists a constant $C = C(M) > 0$ such that $A(p) \leq CI$ for all $p \in \mathbb{R}^n$ satisfying $|p| \leq M$. We also assume a lower bound such that there exists $c > 0$ satisfying $cI \leq A(p)$ for all $p \in \mathbb{R}^n$. For (2.6) $F = F(p, X)$ is given by

$$F(p, X) = -\operatorname{trace} \{ A(p) R_p X R_p \}, \quad R_p := I - \frac{p \otimes p}{1 + |p|^2}. \quad (2.7)$$

This $F = F(p, X)$ is also uniformly elliptic. Elliptic constants are taken by $\lambda_M = c/(1 + M^2)^2$, $\Lambda_M = C$.

3. Strong Comparison Principle

Let Ω be a domain in \mathbb{R}^n . We consider a elliptic equation of the form

$$F(Du(x), D^2u(x)) = 0 \quad \text{in } \Omega. \quad (3.1)$$

Our main theorem is an extension of the strong comparison theorem to viscosity subsolutions and supersolutions to (3.1).

Theorem 3.1. *Suppose that Ω is a domain in \mathbb{R}^n . Assume that F satisfies (F1)–(F3). Let $u \in USC(\Omega)$ and $v \in LSC(\Omega)$ be, respectively, viscosity sub- and supersolutions of (3.1). Assume that $u \leq v$ in Ω and that there exists a point $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$. Then $u \equiv v$ in Ω .*

If v is a constant function in Ω and a constant function is a viscosity solution then Theorem 3.1 gives a strong maximum principle.

We shall prove Theorem 3.1 in several steps. Our proof reflects that of the maximum principle to uniformly elliptic equations in classical sense. Choice of an auxiliary function and some domains in Ω near the point x_0 are very similar to the classical work [9], [5].

Let $a \in \Omega$, $R > 0$,

$$\begin{aligned} B_0 &:= B(a, R) \subset\subset \Omega, \quad x_0 \in \partial B_0, \\ B_1 &:= B(x_0, \frac{R}{2}) \subset\subset \Omega, \end{aligned}$$

where $B(a, R)$ denotes the open ball in \mathbb{R}^n of radius R centered at a . Let for $\gamma > 0$ and $x \in \Omega$

$$z(x) := e^{-\gamma R^2} - e^{-\gamma|x-a|^2}.$$

By definition one observes that

$$\begin{aligned} -1 < z(x) < 0 & \quad \text{in } B_0, \\ z(x) = 0 & \quad \text{on } \partial B_0, \\ 0 < z(x) < 1 & \quad \text{outside } \overline{B_0}. \end{aligned} \tag{3.2}$$

Let $w(x, y)$ be a function on $\Omega \times \Omega$. We set for $(x, y) \in \Omega \times \Omega$ and $\varepsilon, \alpha > 0$,

$$\begin{aligned} \Phi(x, y) &:= \varepsilon z(x) + \alpha|x - y|^2, \\ \Psi(x, y) &:= w(x, y) - \Phi(x, y). \end{aligned}$$

For proof of Theorem 3.1 we have to study maximum points of $\Psi(x, y)$ on $\overline{B_1} \times \overline{B_1}$ and their values. First we shall consider the value of $\Psi(x, x)$ for $x \in \partial B_1$.

Proposition 3.2. *Let B_0, B_1 and $z(x)$ as stated above. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then*

$$w(x, x) - \varepsilon z(x) < 0 \quad \text{on } \partial B_1$$

for all $\gamma > 0$ provided that w is upper semicontinuous on $\Omega \times \Omega$, $w(x, x) \leq 0$ for all $x \in \Omega$ and

$$\begin{cases} w(x, x) < 0 & \text{if } x \in \overline{B_0} \setminus \{x_0\}, \\ w(x_0, x_0) = 0. \end{cases}$$

Proof. We will divide the boundary of B_1 into two pieces:

$$C_1 := \partial B_1 \cap \overline{B_0}, \quad C_2 := \partial B_1 \setminus \overline{B_0};$$

clearly ∂B_1 is a disjoint union of C_1 and C_2 . Since $w(x, x) < 0$ on a compact set C_1 , there exists a constant $\ell > 0$ that satisfies $w(x, x) \leq -\ell$ on C_1 by upper semicontinuity of w . By (3.2) $z(x) \leq 0$ on C_1 . We shall take $\varepsilon_0 > 0$ such that

$$-\ell - \varepsilon_0 z(x) < 0 \quad \text{on } C_1.$$

By (3.2) $z(x) > 0$ on C_2 . We easily see that for any $\varepsilon > 0$

$$w(x, x) - \varepsilon z(x) < 0 \quad \text{on } C_2.$$

Thus we observe that if $0 < \varepsilon < \varepsilon_0$,

$$w(x, x) - \varepsilon z(x) < 0 \quad \text{on } \partial B_1$$

for all $\gamma > 0$. □

We next study properties of maximum points of $\Psi(x, y)$ on $\overline{B_1} \times \overline{B_1}$.

Proposition 3.3. *Suppose that w be upper semicontinuous on $\Omega \times \Omega$ and that*

$$\begin{aligned} w(x, x) &< 0 & \text{if } x \in \overline{B_0} \setminus \{x_0\}, \\ w(x_0, x_0) &= 0. \end{aligned}$$

Let B_0, B_1 and Ψ as stated above and let ε_0 be as in Proposition 3.2. Let $\Psi(x, y)$ attain its maximum at $(x_\alpha, y_\alpha) \in \overline{B_1} \times \overline{B_1}$ for all $0 < \varepsilon < \varepsilon_0$. Then $|x_\alpha - y_\alpha| \rightarrow 0$ as $\alpha \rightarrow +\infty$; this convergence is uniform in $0 < \varepsilon < \varepsilon_0$ and $\gamma > 0$.

In particular, there exists a point $\hat{x} \in \overline{B_1}$ such that $x_\alpha, y_\alpha \rightarrow \hat{x}$ as $\alpha \rightarrow +\infty$ by taking a subsequence.

Proof. We easily see $\Psi(x_\alpha, y_\alpha) \geq 0$ for all $0 < \varepsilon < \varepsilon_0$ and $\gamma > 0$, since $\Psi(x_\alpha, y_\alpha) \geq \Psi(x_0, x_0) = 0$. We observe that

$$w(x_\alpha, y_\alpha) - \varepsilon z(x_\alpha) \geq \alpha |x_\alpha - y_\alpha|^2.$$

By boundedness of $\overline{B_1}$ and upper semicontinuity of w , there exists a positive constant M such that

$$w(x_\alpha, y_\alpha) - \varepsilon z(x_\alpha) \leq M \quad \text{for all } (x_\alpha, y_\alpha) \in \overline{B_1} \times \overline{B_1}.$$

We now observe that

$$0 \leq \alpha |x_\alpha - y_\alpha|^2 \leq M,$$

which yields $|x_\alpha - y_\alpha| \rightarrow 0$ as $\alpha \rightarrow +\infty$. □

Proposition 3.4. *Assume the same hypotheses of Proposition 3.3. Then there exists $\alpha_0 > 0$ such that if $\alpha > \alpha_0$ then Ψ attains its maximum over $\overline{B_1}$ at an interior point $(x_\alpha, y_\alpha) \in B_1 \times B_1$ for all $0 < \varepsilon < \varepsilon_0$ and $\gamma > 0$.*

Proof. We will show $\hat{x} \in B_1$. Suppose that $\hat{x} \in \partial B_1$. By definition of Ψ and $\Psi(x_\alpha, y_\alpha) \geq 0$ we have

$$w(x_\alpha, y_\alpha) - \varepsilon z(x_\alpha) \geq \Psi(x_\alpha, y_\alpha) \geq 0.$$

Letting $\alpha \rightarrow +\infty$ by taking a subsequence we observe that

$$w(\hat{x}, \hat{x}) - \varepsilon z(\hat{x}) \geq 0$$

which contradicts to Proposition 3.2. Thus if $\alpha > 0$ is sufficiently large say $\alpha > \alpha_0$, then $x_\alpha, y_\alpha \in B_1$. □

For the proof of Theorem 3.1 we will use a maximum principle for semicontinuous functions due to Crandall and Ishii [3]. In particular, we shall study several properties on matrices which are useful to calculate matrices appeared in their theory.

Let

$$\begin{aligned} d(x, \gamma) &:= 2\varepsilon\gamma e^{-\gamma|x-a|^2}, \\ B &:= d(x, \gamma)(I - 2\gamma(x - a) \otimes (x - a)). \end{aligned}$$

Lemma 3.5. *For all $0 < \lambda \leq 1$, $0 < \varepsilon \leq 1$ and $N_1 > 0$ there exists $\gamma_0 > 0$ such that if $\gamma > \gamma_0$, then:*

$$B + \lambda B^2 \leq 2d(x, \gamma)I, \tag{i}$$

$${}^t\nu(B + \lambda B^2)\nu \leq -d(x, \gamma)|\nu|^2 N_1 \quad \text{for all } x \in B_1, \quad (\text{ii})$$

where ν is an outward normal vector on ∂B_0 at $x_0 \in \partial B_0$ such that $\nu = x_0 - a$.

Proof. (i) By direct calculation we have

$$\begin{aligned} B + \lambda B^2 &= d(x, \gamma)[I - 2\gamma(x - a) \otimes (x - a) \\ &\quad + \lambda d(x, \gamma)\{I - 4\gamma(x - a) \otimes (x - a) + 4\gamma^2|x - a|^2(x - a) \otimes (x - a)\}] \\ &\leq d(x, \gamma)[I + \lambda d(x, \gamma)\{I + 4\gamma^2|x - a|^2(x - a) \otimes (x - a)\}]. \end{aligned}$$

We set

$$M := d(x, \gamma)\{I + 4\gamma^2|x - a|^2(x - a) \otimes (x - a)\}.$$

We note that there exists $\gamma_1 > 0$ such that if $\gamma > \gamma_1$ then $M \leq I$.

$$\begin{aligned} M - I &= (d(x, \gamma) - 1)I + 4\gamma^2 d(x, \gamma)|x - a|^2(x - a) \otimes (x - a), \\ {}^t p(M - I)p &= (d(x, \gamma) - 1)|p|^2 + 4\gamma^2 d(x, \gamma)|x - a|^2 \langle p, x - a \rangle^2 \\ &\quad \text{for all } p \in \mathbb{R}^n. \end{aligned}$$

Since $\frac{R}{2} \leq |x - a| \leq \frac{3}{2}R$, we have

$$d(x, \gamma) \rightarrow 0 \quad \text{and} \quad \gamma^2 d(x, \gamma) \rightarrow 0 \quad \text{as } \gamma \rightarrow +\infty.$$

Now we observe that there exists $\gamma_1 > 0$ so that

$$(d(x, \gamma) - 1)|p|^2 + 4\gamma^2 d(x, \gamma)|x - a|^2 \langle p, x - a \rangle^2 \leq 0$$

if $\gamma > \gamma_1$. Therefore, we have if $\gamma > \gamma_1$ then $M \leq I$. By $\lambda \leq 1$ we see that if $\gamma > \gamma_1$, then

$$B + \lambda B^2 \leq 2d(x, \gamma)I \quad \text{for all } x \in B_1.$$

(ii) By direct calculation and the Schwarz inequality

$$\begin{aligned} {}^t\nu(B + \lambda B^2)\nu &= d(x, \gamma)[|\nu|^2 - 2\gamma \langle \nu, x - a \rangle^2 \\ &\quad + \lambda d(x, \gamma)(|\nu|^2 - 4\gamma \langle \nu, x - a \rangle^2 + 4\gamma^2|x - a|^2 \langle \nu, x - a \rangle^2)] \\ &\leq d(x, \gamma)[|\nu|^2 - 2\gamma \langle \nu, x - a \rangle^2 + \lambda d(x, \gamma)(|\nu|^2 + 4\gamma^2|\nu|^2|x - a|^4)] \\ &= d(x, \gamma)|\nu|^2[1 - 2\gamma \langle \frac{\nu}{|\nu|}, x - a \rangle^2 + \lambda d(x, \gamma)(1 + 4\gamma^2|x - a|^4)]. \end{aligned}$$

Note that $\langle \nu, x - a \rangle > 0$ for all $x \in B_1$. For all $N_1 > 0$ there exists $\gamma_2 > 0$ such that if $\gamma > \gamma_2$ then

$$1 - 2\gamma \langle \frac{\nu}{|\nu|}, x - a \rangle^2 + \lambda d(x, \gamma)(1 + 4\gamma^2|x - a|^4) \leq -N_1$$

for all $x \in B_1$. Thus for all $N_1 > 0$ there exists $\gamma_2 > 0$ such that if $\gamma > \gamma_2$ then

$${}^t\nu(B + \lambda B^2)\nu \leq -d(x, \gamma)|\nu|^2 N_1 \quad \text{for all } x \in B_1x \quad \square$$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We will argue by contradiction. We set $w(x, y) = u(x) - v(y)$ so that w is upper semicontinuous on $\Omega \times \Omega$. Suppose that there would exist a point $x_1 \in \Omega$ such that $u(x_1) < v(x_1)$. By a standard argument there would exist an open ball B_0 with $\overline{B_0} \subset \Omega$ and $x'_0 \in \partial B_0$ that satisfies

$$\begin{aligned} u < v & \quad \text{in } \overline{B_0} \setminus \{x'_0\}, \\ u(x'_0) & = v(x'_0). \end{aligned}$$

We shall replace x'_0 with x_0 since $u(x_0) = v(x_0)$. We set $B_0 = B(a, R)$ and $B_1 = B(x_0, \frac{R}{2})$ so that $\overline{B_1} \subset \Omega$. Now we see that all conclusions of Proposition 3.2 – Proposition 3.4 would hold for $\Psi = w - \Phi$ on $\overline{B_1} \times \overline{B_1}$ for sufficiently small ε and sufficiently large α . Proposition 3.4 says that Ψ attains its maximum over $\overline{B_1} \times \overline{B_1}$ at $(x_\alpha, y_\alpha) \in B_1 \times B_1$ for sufficiently small $\varepsilon > 0$ and sufficiently large $\alpha > 0$. In particular,

$$u(x) - v(y) \leq u(x_\alpha) - v(y_\alpha) + \Phi(x, y) - \Phi(x_\alpha, y_\alpha).$$

Expanding Φ at (x_α, y_α) we get

$$\left(\begin{pmatrix} \Phi_x(x_\alpha, y_\alpha) \\ \Phi_y(x_\alpha, y_\alpha) \end{pmatrix}, A \right) \in J^{2,+}(u(x_\alpha) - v(y_\alpha))$$

with

$$A = D^2\Phi(x_\alpha, y_\alpha) = \begin{pmatrix} \Phi_{xx}(x_\alpha, y_\alpha) & \Phi_{yx}(x_\alpha, y_\alpha) \\ \Phi_{xy}(x_\alpha, y_\alpha) & \Phi_{yy}(x_\alpha, y_\alpha) \end{pmatrix},$$

where $\Phi_x = D_x\Phi$, $\Phi_{xx} = D_{xx}^2\Phi$ is an $n \times n$ matrix and so on. We shall apply the elliptic version of Crandall–Ishii’s Lemma [3, Theorem 3.2]. We see that for all positive λ , there exists $X, Y \in \mathbb{S}^n$ such that

(i)

$$\begin{aligned} (\Phi_x(x_\alpha, y_\alpha), X) & \in \overline{J^{2,+}}u(x_\alpha), \\ (\Phi_y(x_\alpha, y_\alpha), Y) & \in \overline{J^{2,+}}(-v(y_\alpha)) \\ (\Leftrightarrow) \quad (-\Phi_y(x_\alpha, y_\alpha), -Y) & \in \overline{J^{2,-}}v(y_\alpha), \end{aligned}$$

(ii)

$$-\left(\frac{1}{\lambda} + \|A\|\right) I_{2n} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq A + \lambda A^2. \tag{MI}$$

Here $\overline{J^{2,+}}$ and $\overline{J^{2,-}}$, respectively, denote closure of $J^{2,+}$ and $J^{2,-}$ (cf. [3]).
By direct calculation

$$\begin{aligned}\Phi_x &= \varepsilon D z(x) + 2\alpha(x - y) = 2\varepsilon\gamma e^{-\gamma|x-a|^2}(x - a) + 2\alpha(x - y), \\ \Phi_y &= -2\alpha(x - y), \\ \Phi_{xx} &= \varepsilon D^2 z(x) + 2\alpha I = 2\varepsilon\gamma e^{-\gamma|x-a|^2}(I - 2\gamma(x - a) \otimes (x - a)) + 2\alpha I, \\ \Phi_{xy} &= \Phi_{yx} = -2\alpha I, \quad \Phi_{yy} = 2\alpha I.\end{aligned}$$

By definition of d and B (see the paragraph just before Lemma 3.5) we obtain the identity at $x = x_\alpha$

$$A = \begin{pmatrix} B + 2\alpha I & -2\alpha I \\ -2\alpha I & 2\alpha I \end{pmatrix}.$$

From (MI) we observe that

$$X + Y \leq B + \lambda B^2.$$

Let $\rho(x, \gamma) = d(x, \gamma)(x - a)$ and let $p_\alpha = 2\alpha(x - y)$ so that $\Phi_x = \rho(x, \gamma) + p_\alpha$. Since u is a viscosity subsolution of (3.1), we have

$$F(\rho(x_\alpha, \gamma) + p_\alpha, X) \leq 0. \quad (3.3)$$

Since v is a viscosity supersolution of (3.1), we have

$$F(p_\alpha, -Y) \geq 0. \quad (3.4)$$

Subtracting (3.4) from (3.3), we get

$$F(\rho(x_\alpha, \gamma) + p_\alpha, X) - F(p_\alpha, -Y) \leq 0. \quad (3.5)$$

By (F3) we see that

$$\begin{aligned}F(\rho(x_\alpha, \gamma) + p_\alpha, X) - F(p_\alpha, X) &\geq -L_{M,K}|\rho(x_\alpha, \gamma)| \\ &= -L_{M,K}d(x_\alpha, \gamma)|x_\alpha - a|.\end{aligned}$$

By (F2) and Lemma 3.5 we observe that

$$F(p_\alpha, X) - F(p_\alpha, -Y) \geq f(p_\alpha, X + Y) \geq 2d(x_\alpha, \gamma)\beta\left(\frac{N_1}{2}\right)$$

for all $N_1 > 0$ by taking γ sufficiently large. From (3.5) and $R \leq 2|x_\alpha - a| \leq 3R$ we see

$$0 \geq 2d(x_\alpha, \gamma)\beta(N_2) - L_{M,K}d(x_\alpha, \gamma)\frac{3}{2}R,$$

where $N_1 = 2N_2$. Since $d(x, \gamma) > 0$ we have

$$0 \geq 2\beta(N_2) - L_{M,K} \frac{3}{2}R.$$

Letting $N_2 \rightarrow +\infty$ yields $\beta(N_2) \rightarrow +\infty$. This means that there exists N_0 such that if $N_2 > N_0$ then

$$L_{M,K} \frac{3}{2}R < 2\beta(N_2).$$

We get a contradiction. Now we have completed the proof of Theorem 3.1. \square

Remark 3.6. Our Theorem 3.1 can apply a equation which depends on a space variable x of the form

$$F(Du(x), D^2u(x)) - \varphi(x) = 0 \quad \text{in } \Omega, \tag{3.6}$$

provided $\varphi \in C(\Omega)$. The proof is almost same as that of Theorem 3.1. By the same procedure we have

$$0 \geq 2d(x_\alpha, \gamma)\beta(N_2) - L_{M,K}d(x_\alpha, \gamma)\frac{3}{2}R - \varphi(x_\alpha) + \varphi(y_\alpha).$$

From Proposition 3.2 letting $\alpha \rightarrow +\infty$ by taking a subsequence we observe that

$$0 \geq 2d(\hat{x}, \gamma)\beta(N_2) - L_{M,K}d(\hat{x}, \gamma)\frac{3}{2}R$$

for some $\hat{x} \in B_1$. Since $d(\hat{x}, \gamma) > 0$ we get a contradiction again. A typical example of (3.6) is

$$-\text{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = \varphi(x) \quad \text{in } \Omega,$$

which is called the curvature equation.

4. The Hopf Boundary Lemma

In this section we establish the Hopf Boundary Lemma. We consider a equation of the form

$$F(Du, D^2u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{4.1}$$

Theorem 4.1. (The Hopf Boundary Lemma) *Suppose that F satisfies (F1), (F2) and (F3). Let $u \in \text{USC}(\Omega \cup \{x_0\})$ and $v \in \text{LSC}(\Omega \cup \{x_0\})$ be a viscosity subsolution and a supersolution of (4.1), respectively.*

Assume that

$$u \leq v \quad \text{in } \Omega \cup \{x_0\},$$

there exists a ball $B_0 \subset \Omega$ and a point $x_0 \in \partial B_0$ such that

$$u < v \quad \text{in } \bar{B}_0 \setminus \{x_0\}$$

and $u(x_0) = v(x_0)$.

Then for any $w \in \mathbb{R}^n$ satisfying $\langle w, \nu \rangle < 0$,

$$\limsup_{s \downarrow 0} \frac{(u-v)(x_0+sw) - (u-v)(x_0)}{s} \leq c \langle w, \nu \rangle \quad (4.2)$$

with some $c > 0$ independent of w and ν , where ν denotes the outward normal of the boundary ∂B at x_0 .

Proof. Let $B_0 = B(a, R)$ and let z be the same function as in (3.2).

To show (4.2) it suffices to prove

$$(u - v - \varepsilon z)(x) \leq 0 \quad \text{in } Z \quad (4.3)$$

for sufficiently small $\varepsilon > 0$ ($0 < \varepsilon < 1$) and a domain Z which is neighborhood of x_0 and is contained in B_0 . If we have (4.3), we can see

$$(u - v - \varepsilon z)(x_1) \leq (u - v - \varepsilon z)(x_0) \quad \text{for all } x_1 \in Z.$$

For small $s > 0$ we set $x_1 = x_0 + sw$. Now we observe that

$$\frac{(u-v)(x_0+sw) - (u-v)(x_0)}{s} \leq \frac{\varepsilon z(x_0+sw) - \varepsilon z(x_0)}{s}.$$

Since $\langle \nu, w \rangle < 0$, we get

$$\begin{aligned} \limsup_{s \downarrow 0} \frac{(u-v)(x_0+sw) - (u-v)(x_0)}{s} &\leq \varepsilon \langle Dz(x_0), w \rangle \\ &= 2\varepsilon \gamma e^{-\gamma R^2} \langle \nu, w \rangle < 0. \end{aligned}$$

Thus we obtain (4.2).

It remains to prove (4.3). We argue by contradiction. Let $B_1 = B(x_0, \frac{R}{2})$ and $Z = B_0 \cap B_1$. Suppose that for all ε ($0 < \varepsilon < 1$) there would exist $\tilde{x} \in \bar{Z}$ such that

$$(u - v - \varepsilon z)(\tilde{x}) = \max_{\bar{Z}}(u - v - \varepsilon z) = \sigma_\varepsilon > 0.$$

On the boundary ∂Z there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then

$$(u - v - \varepsilon z)(x) \leq 0 \quad \text{on} \quad \partial Z. \tag{4.4}$$

We see that $\tilde{x} \in Z$ and

$$\max_Z (u - v - \varepsilon z) = \sigma_\varepsilon.$$

Now we set

$$\Phi(x, y) = \varepsilon z(x) + \alpha |x - y|^2,$$

where $\alpha > 0$. We define

$$\Psi(x, y) = u(x) - v(y) - \Phi(x, y).$$

Let Ψ attain its maximum at $(\bar{x}, \bar{y}) \in \bar{Z} \times \bar{Z}$ for all $\varepsilon \in (0, \varepsilon_0)$ and $\alpha > 0$, i.e.,

$$\max_{\bar{Z} \times \bar{Z}} \Psi(x, y) = \Psi(\bar{x}, \bar{y}).$$

We easily see that $\Psi(\bar{x}, \bar{y}) > 0$ since

$$\max_{\bar{Z} \times \bar{Z}} \Psi(x, y) \geq \max_Z (u - v - \varepsilon)(x) = \sigma_\varepsilon > 0. \tag{4.5}$$

We observe that

$$M \geq u(\bar{x}) - v(\bar{y}) - \varepsilon z(\bar{x}) > \alpha |\bar{x} - \bar{y}|^2 \geq 0$$

and there exists $\hat{x} \in \bar{Z}$ such that

$$\bar{x}, \bar{y} \rightarrow \hat{x} \quad \text{as} \quad \alpha \rightarrow +\infty$$

by taking a subsequence. Note that $\hat{x} \in Z$. Suppose that $\hat{x} \in \partial Z$. By (4.5)

$$u(\bar{x}) - v(\bar{y}) - \varepsilon z(\bar{x}) \geq u(\bar{x}) - v(\bar{y}) - \varepsilon z(\bar{x}) - \alpha |\bar{x} - \bar{y}|^2 \geq \sigma_\varepsilon > 0.$$

Letting $\alpha \rightarrow +\infty$ by taking a subsequence we have $(u - v - \varepsilon z)(\hat{x}) > 0$ that contradicts (4.4). Thus if $\alpha > 0$ is sufficiently large say $\alpha > \alpha_0$, then $\bar{x}, \bar{y} \in Z$. Since $u(x) - v(y) \leq u(\bar{x}) - v(\bar{y}) + \Phi(x, y) - \Phi(\bar{x}, \bar{y})$, we argue in the same way as in the proof of Theorem 3.1 with $x_\alpha = \bar{x}$, $y_\alpha = \bar{y}$ to get a contradiction. \square

Remark 4.2. Our result roughly speaking that $\partial u / \partial \nu < \partial v / \partial \nu$ at $x = x_0$ if u and v are differentiable at $x = x_0$. For linear elliptic equations the Hopf Boundary Lemma implies the strong maximum principle. For some nonlinear degenerate elliptic equations a version of the Hopf Boundary Lemma is established by [1, Theorem 1] to prove the strong maximum principle for semi-continuous viscosity solutions. In their situation v is taken a constant.

The proof of Theorem 4.1 is essentially the same as that of Theorem 3.1. However, u and v may not satisfies the equation (4.1) at $x = x_0$. So we should discuss separately the place where $w - \Phi$ takes maximum values.

5. Key Lemma for Uniformly Elliptic Equations

We give some examples of equation (1.1) and we shall check (F2) holds. Our condition (F2) holds for uniformly elliptic equations (cf. Proposition 2.4 and 2.5). To verify $-\Lambda \text{trace}(X + Y)_+ - \lambda \text{trace}(X + Y)_-$ which is appeared in Proposition 2.4 is a coercive function, we prepare the following lemma.

Lemma 5.1. *Let $\Lambda \geq \lambda > 0$. Suppose that $b > 0$, $N > 0$, $S \in \mathbb{S}^n$ satisfy*

$$S \leq bI, \quad (5.1)$$

$${}^t\mu S \mu \leq -bN \quad \text{for some } \mu \in S^{n-1}, \quad (5.2)$$

where S^{n-1} denotes the set of unit vector in \mathbb{R}^n . Then we have

$$\Lambda \text{trace} S_+ + \lambda \text{trace} S_- \leq \Lambda(n-1)b - \frac{\lambda N}{n}b.$$

Proof. We may assume that S is a diagonal matrix. Let λ_i ($1 \leq i \leq n$) be eigenvalues of S . From (5.1) we see $\lambda_i \leq b$ for all i . From (5.2) there exists number ℓ that satisfies $\lambda_\ell \leq -bN/n$. We may assume that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \geq 0 > \lambda_{j+1} \geq \cdots \geq \lambda_{n-1} \geq \lambda_n.$$

From (5.2) at least one eigenvalue is negative. We do not worry about the case all eigenvalues are negative. By the definition of S_+ and S_- we see that

$$\text{trace} S_+ = \sum_{k=1}^j \lambda_k, \quad \text{trace} S_- = \sum_{k=j+1}^n \lambda_k.$$

Then we obtain

$$\Lambda \text{trace} S_+ + \lambda \text{trace} S_- = \Lambda \sum_{k=1}^j \lambda_k + \lambda \sum_{k=j+1, k \neq \ell}^n \lambda_k + \lambda \lambda_\ell.$$

By (5.1) and (5.2) we see that

$$\leq \Lambda \sum_{k=1}^j b + \lambda \sum_{k=j+1, k \neq \ell}^n b - \lambda \frac{Nb}{n} \leq \Lambda(n-1)b - \lambda \frac{Nb}{n}. \quad \square$$

Remark 5.2. By Proposition 2.4 and Lemma 5.1 we conclude that to uniformly elliptic equations coercive function f and a function β which is a bound for f are following; for each $M > 0$ if $|p| \leq M$ then

$$f(p, S) = -\Lambda_M \text{trace} S_+ - \lambda_M \text{trace} S_-,$$

$$\beta(N) = -\Lambda_M(n-1) + \frac{\lambda_M N}{n}.$$

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