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SOME NONLINEAR ERGODICS THEOREMS ON HILBERT SPACES

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1. Introduction

In [1] Baillon has shown the first nonlinear ergodic theorem for nonexpansive mapping. Recently Kakutani et Rouhani [3] studied the problem of the convergence the Cesàro means of contractive sequence. Later, Oukili proved in [4] that there is an equivalent between the two works and studied the problem of the convergence the means of contractive family.

In this paper we establish several results involving hypotheses weak enough to include a number of ergodic theorems as special cases. Furthermore, we deal with the problem relative to nonlinear ergodic theory for the mappings and semigroups are not necessarily continuous.

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Throughout this paper, let C be a nonempty convex subset of a Hilbert space H endowed with the scalar product $<,>$, and $w-\lim$ denote the weak limit. We introduce the following definition.

Definition 1. A mapping $T : C \to C$ is called (α)-mapping on C, if there are two constants a, b such that $a + 2b = 1$ and

$$\| Tx - Ty \| \leq a \| x - y \| + b[\| Tx - y \| + \| Ty - x \|]$$

for all $x,\ y \in C$.

Note that a nonexpansive mapping is (α)-mapping with $b = 0$. But the converse is not true as is shown in the following example

Example 2. Define $T : [0, 1] \to [0, 1]$ as follows

$$Tx = \begin{cases} o \text{ if } x = 1\,, \\ \frac{1}{2} \text{ if } x \neq 1\,. \end{cases}$$

It is easy to see that T is not nonexpansive for any norm defined on R. But T is (α)-mapping with $a = b = \frac{1}{3}$.

Likewise, we have the folowing definition.

Definition 3. A sequence $\{a(n)\} \subset C$ (resp. a family $(a(t),\ t \geq 0)$ of elements of C) is called (α)-sequence (resp.(α)- family), if there are two constants a, b such that $a + 2b = 1$ and

$$\| a(n+1) - a(m+1) \| \\ \leq a \| a(n) - a(m) \| + b[\| a(n+1) - a(m) \| + \| a(m+1) - a(n) \|]\,,$$

for all $n,\ m \in N$ (respectively

$$\| a(v+e) - a(u+e) \| \\ \leq a \| a(v) - a(u) \| + b[\| a(u+e) - a(v) \| + \| a(v+e) - a(u) \|]\,,$$

for all $u,\ v\text{et } e \in R^+$).

2. Main Results

Theorem 4. *Let $(a(n), n \in N)$ be a bounded (α)-sequence in H. Then*

$$S(n) = \frac{1}{n} \sum_{i=0}^{n-1} a(i)$$

converge weakly in H.

Proof. Since $(S(n),\ n \in N)$ is a bounded sequence in Hilbert space, then there exists un subsequence $(n(k),\ k \in N)$ in N converges to ∞ as $k \to \infty$ and an element x in H such that:

$$w - \lim_{k\to\infty} S(n(k)) = x.$$

Assume that there exists another subsequence $(v(k), k \in N)$ in N, $v(k) \to \infty$ as $k \to \infty$ and an element y in H such that

$$w - \lim_{k\to\infty} S(v(k)) = y.$$

So, to complete the proof it suffices to show that $y = x$.

Similarly, we can suppose that:

$$\begin{cases} w - \lim\limits_{k\to\infty} S(n(k)) = x\,, \\ \dfrac{1}{n(k)} \displaystyle\sum_{i=0}^{n(k)-1} \parallel a(i) \parallel^2 = z\,, \\ \dfrac{1}{n(k)} \displaystyle\sum_{i=0}^{n(k)-1} \parallel a(i) - a(s) \parallel^2 = f(s) \quad \text{for all } s \in N\,, \end{cases}$$

and

$$\begin{cases} w - \lim\limits_{k\to\infty} S(v(k)) = y\,, \\ \dfrac{1}{v(k)} \displaystyle\sum_{i=0}^{v(k)-1} \parallel a(i) \parallel^2 = w\,, \\ \dfrac{1}{v(k)} \displaystyle\sum_{i=0}^{v(k)-1} \parallel a(i) - a(s) \parallel^2 = g(s) \text{ for all } s \in N\,. \end{cases}$$

Since $(a(n), n \in N)$ is a (α)-sequence then f and g are decreasing functions, we show for example that f is decreasing function. Indeed

$$\begin{aligned} \parallel a(i+1) - a(s+1) \parallel^2 &\le a^2 \parallel a(i) - a(s) \parallel^2 + b^2 \parallel a(i) - a(s+1) \parallel^2 \\ &\quad + b^2 \parallel a(i+1) - a(s) \parallel^2 + 2ab \parallel a(i) - a(s) \parallel [\parallel a(i+1) - a(s) \parallel \\ &\quad + \parallel a(i) - a(s+1) \parallel] + b^2 \parallel a(i) - a(s+1) \parallel \parallel a(i+1) - a(s) \parallel \\ \le [a^2+2ab] \parallel a(i)-a(s) \parallel^2 &+ [2b^2+ab][\parallel a(i+1)-a(s) \parallel^2 + \parallel a(i)-a(s+1) \parallel^2]\,. \end{aligned}$$

Then

$$\frac{1}{n(k)}\sum_{i=0}^{n(k)-1}\parallel a(i+1)-a(s+1)\parallel^2\leq[a^2+2ab]\frac{1}{n(k)}\sum_{i=0}^{n(k)-1}\parallel a(i)-a(s)\parallel^2$$
$$+[2b^2+ab]\frac{1}{n(k)}\sum_{i=0}^{n(k)-1}\parallel a(i+1)-a(s)\parallel^2$$
$$+[2b^2+ab]\frac{1}{n(k)}\sum_{i=0}^{n(k)-1}\parallel a(i)-a(s+1)\parallel^2\ .$$

Letting $k\to\infty$, we obtain

$$f(s+1)\leq(a^2+2ab)f(s)+(2b^2+ab)[f(s)+f(s+1)].$$

Since f is positive, decreasing function. Then we have

$$p=\inf\{f(s),\ s\in N\},\quad\text{and}\ \ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}f(i)=p.$$

So, using the parallelogram legality in H, we get

$$< a(i),a(j)>=\frac{1}{2}[\parallel a(i)\parallel^2+\parallel a(j)\parallel^2-\parallel a(i)-a(j)\parallel^2],$$

for all $i,\ j\in N$. Next

$$< S(n),S(v)>=\frac{1}{2}[\frac{1}{n}\sum_{i=0}^{n-1}\parallel a(i)\parallel^2+\frac{1}{v}\sum_{j=0}^{v-1}\parallel a(j)\parallel^2$$
$$-\frac{1}{n}\frac{1}{v}\sum_{i=0}^{n-1}\sum_{j=0}^{v-1}\parallel a(i)-a(j)\parallel^2]\,,\quad(1)$$

for all $n,\ v\in N$.

We replace in (1), n by $n(k)$ and, letting as $k\to\infty$, we obtain

$$< x,S(v)>=\frac{1}{2}[z+\frac{1}{v}\sum_{j=0}^{v-1}\parallel a(j)\parallel^2-\frac{1}{v}\sum_{j=0}^{v-1}f(j)]\,.\qquad(2)$$

Similary, we replace in (2), v by $n(k)$ and, letting $k\to\infty$, we get

$$< x,x>=\frac{1}{2}[z+z-p].\qquad(3)$$

Using (2), we replace v by $v(k)$. As $k \to \infty$, we obtain

$$< x, y >= \frac{1}{2}[z + w - p]. \tag{4}$$

By symmetry, $q = \inf\{g(s),\ s \in N\}$ we can write

$$< y, y >= \frac{1}{2}[w + w - q], \tag{5}$$

$$< y, x >= \frac{1}{2}[w + z - q]. \tag{6}$$

So from equations (3), (4), (5), (6) and the following identity

$$< x - y, x - y >=< x, x > + < y, y > -2 < x, y >,$$

we have

$$< x - y, x - y >= 0.$$

This completes the proof of the theorem. □

Theorem 5. *Let $(a(t),\ t \geq 0)$ be a bounded (α)- family in H such that $a :\ R^+ \to H$ is a continuous mappings. Then*

$$S(t) = \frac{1}{t}\int_0^t a(u)\, du$$

weakly converge in H as t tend to ∞.

The proof of above theorem is straightforward analogue of proof of the Theorem 4.

2.1. Applications

Let $\Gamma = (T(t),\ t \geq 0)$ be a family of (α)-mappings ($a,\ b$ independent of t) from C to C, Γ is called strongly continuous semigroup of C, if:

1) For each $x \in C,\ t \to T(t)x$ is continuous on R^+,

2) $T(t_1 + t_2) = T(t_1) \circ T(t_2)$ for all $t_1,\ t_2 \geq 0$, and

3) For all $x \in C$

$$\lim_{t \to 0^+} \| T(t)x - x \|= 0.$$

Our next corollary is an extension of a Baillon's Theorem, see [1].

Corollary 6. *Let $T : C \to C$ be a (α)-mapping with C be a closed convex subset a Hilbert space, if there exists an element x_0 of C such that $(T^i x_0,\ i \in N)$ is bounded. Then for all $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$$

weakly converge to a fixed point of T in C.

Proof. From Theorem 4, let

$$y = w - \lim_{n\to\infty} S_n x_0,$$

which implies, since C is weakly closed (by using the fact that C closed convex subset) that $y \in C$. The other parts since T is (α)-mapping, then

$$\| T^{i+1}x_0 - Ty \|^2 \le \tilde{a} \| T^i x_0 - y \|^2 + \tilde{b}[\| T^{i+1}x_0 - y \|^2 + \| Ty - T^i x_0 \|^2], \quad \text{(A)}$$

with $\tilde{a} = a^2 + 2ab$ and $\tilde{b} = 2b^2 + ab$. So

$$\| T^j x_0 - y \|^2 = \| T^j x_0 - Ty \|^2 + \| y - Ty \|^2 + 2 < T^j x_0 - Ty,\ Ty - y > . \quad \text{(B)}$$

Put $\beta_j = \| T^j x_0 - Ty \|^2$. From (B) and (A) we have

$$\beta_{i+1} \le \tilde{a}[\beta_i + \| Ty - y \|^2 + 2 < T^i x_0 - Ty, Ty - y >] + \tilde{b}[\beta_{i+1} + \| Ty - y \|^2 + 2 < T^{i+1}x_0 - Ty, Ty - y >] + \tilde{b}\beta_i\,.$$

An easy induction shows that

$$(\tilde{a}+\tilde{b})\frac{1}{n}\sum_{i=0}^{n-1}[\beta_{i+1} - \beta_i] \le [\tilde{a}+\tilde{b}] \| Ty - y \|^2 + 2(\tilde{a}+\tilde{b}) < S_n x_0 - Ty, Ty - y > + \frac{2\tilde{b}}{n}[< T^n x_0 - Ty, Ty - y > - < x_0 - Ty, Ty - y >],$$

hence

$$(\tilde{a}+\tilde{b})\frac{1}{n}[\beta_n - \beta_0] \le [\tilde{a}+\tilde{b}] \| Ty - y \|^2 + 2(\tilde{a}+\tilde{b}) < S_n x_0 - Ty, Ty - y >$$

$$+\frac{2\tilde{b}}{n}[< T^n x_0 - Ty, Ty - y > - < x_0 - Ty, Ty - y >]\,.$$

Letting $n \to \infty$, we obtain

$$0 \leq [\tilde{a} + \tilde{b}] \parallel Ty - y \parallel^2 - 2[\tilde{a} + \tilde{b}] \parallel Ty - y \parallel^2\,,$$

this means that

$$\parallel Ty - y \parallel^2 = 0.$$

Then, to complete the proof it suffices to show that for all $x \in C$ $(T^i x,\ i \in N)$ is bounded. So, since T is a (α)-mapping and $y \in$Fix(T); then for all $x \in C$

$$\parallel T^i x - y \parallel \leq \parallel x - y \parallel\,,$$

therefore, $(T^i x,\ i \in N)$ is bounded. □

The following result is a direct extension of a main result in [2] and it can be proved by using the same argument and the methods very similar of the proof of above corollary.

Corollary 7. *Let $\Gamma = (T(t),\ t \geq 0)$ ba a strongly continuous (α)-semigroup of a convex closed subset C of a Hilbert space, if there exists an element x_0 of C such that $(T(t)x_0,\ t \geq 0)$ is bounded. Then for each $x \in C$.*

$$S(t) = \frac{1}{t}\int_0^t T(u)x\,du$$

weakly converge to fixed point y of Γ (i.e. $T(t)y = y$ for all $t \geq 0$) as t tend to ∞.

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