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COMMON FIXED POINT THEOREM FOR (β)-SEMIGROUPS

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Abstract: In the present paper we prove a common fixed theorem for (β)-semigroup in Hilbert spaces. This theorem extend some results of A. Mbarki and al., see [3]

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1. Introduction

There are extensive literature that studied the fixed point for Lipschitzian mappings, for example J.B. Baillon in [1] have shown that, if C is a bounded convex closed subset of a Hilbert space and $T : C \to C$ be a Lipschitzian mapping such that

$$\overline{\lim_n} \parallel T^n \parallel_{Lip} < \sqrt{2},$$

then T has a fixed point in C. Recently in [3] we have established the continuous version of this result by showing the following result.

Theorem 1. *Let C be a bounded convex closed subset of a Hilbert space, and S be a left reversible semigroup of Lipschitzian selfmaps on C such that*

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$$\overline{\lim}_{t\in S}k_t < \sqrt{2}, \tag{1}$$

then S has a fixed point in C.

In this works we will extend Mbarki's Theorem [3] to left reversible (β)-semigroups.

In the sequel we will using the followings lemma and definition.

Lemma 2. (cf. [4]) *Let $(M, ||||)$ be a vectorial space endowed with the norm $||.||$, let S be a left reversible semigroup of selfmaps on M and $x, y \in M$. Then for all $a \in S$ we have*

$$\overline{\lim}_{t\in S} \parallel tx - y \parallel = \overline{\lim}_{t\in S} \parallel atx - y \parallel .$$

Definition 3. Let C be a subset of a Banach space E and S be a of semigroup selfmaps on C. S is called (β)-semigroup, if there are two nets $\{a(t)\}$ and $\{b(t)\}$ of positive real numbers such that $b(t) < 1$ and

$$\parallel S(t)x - S(t)y \parallel \leq a(t) \parallel x - y \parallel + b(t)(\parallel S(t)x - y \parallel + \parallel S(t)y - x \parallel),$$

for all $t \geq 0$ and $x, y \in C$.

Note that all semigroup of Lipschitzian selfmaps is (β)-semigroup with $b = 0$. But the converse is not true as is shown in the following example.

Example 4. Define $T : [0,1] \to [0,1]$ as follows

$$Tx = \begin{cases} 0 \text{ if } x = 1\,, \\ \frac{1}{2} \text{ if } x \neq 1\,. \end{cases}$$

It is easy to see that $S := \{T^n, n = 0, 1..\}$ is not a semigroup of Lipschitzian selfmaps for any norm defined on R. But S is (β)-semigroup with $a_n = b_n = \frac{1}{3}$.

Then as improvement of Mbarki's Theorem [3] we have the following result.

Theorem 5. *Let C be a bounded convex closed subset of a Hilbert space, let S be a left reversible (β)-semigroup of selfmaps on C such that*

$$(1) \quad \overline{\lim_{t}}\left(\frac{a(t) + b(t)}{1 - b(t)}\right) < \sqrt{2}.$$

Then $F(S)$ is nonempty set.

In order to prove Theorem 5 we need the following lemma (for the prove of the lemma see for example [2]).

Lemma 6. (cf. [2]) *Let E be a reflexive Banach space, C be a bounded convex closed subset of E and $\varphi : C \to]-\infty, +\infty]$ be a continuous convex function. Then there exists $x_0 \in C$ such that $\varphi(x_0) = \min\{\varphi(x) \quad x \in C\}$.*

Poof. For $x \in C$, we defined the following function

$$y \longmapsto v_x(y) \quad \text{by} \quad v_x(y) = \overline{\lim}_{t\in S} \parallel tx - y \parallel .$$

It easy to see that v_x is a convex positive function. In other hand, v_x is Lipschitzian, since for $y_1, y_2 \in C$ we have

$$\parallel y_1 - tx \parallel \leq \parallel y_1 - y_2 \parallel + \parallel y_2 - tx \parallel,$$

for all $t \in S$. Taking the limitsup of this as $t \to \infty$, we obtain

$$v_x(y_1) \leq \parallel y_1 - y_2 \parallel + v_x(y_2). \qquad (i)$$

An argument similar we have

$$v_x(y_2) \leq \parallel y_1 - y_2 \parallel + v_x(y_1). \qquad (ii)$$

Then from (i) and (ii) we obtain

$$| v_x(y_1) - v_x(y_2) | \leq \parallel y_1 - y_2 \parallel .$$

Consequently v_x is continuous. Therefore from the Lemma 6 there exists $u \in C$ (which depends of x) such that

$$v_x(u) = \min\{v_x(y) \ : \ y \in C\}.$$

Moreover, u is unique. Indeed, it suffices to show that v_x satisfies the following propriety

$$v_x^2(\alpha_1 y_1 + \alpha_2 y_2) + \alpha_1\alpha_2 \parallel y_1 - y_2 \parallel^2 \leq \alpha_1 v_x^2(y_1) + \alpha_2 v_x^2(y_2)\,, \qquad (2)$$

for all $\alpha_1,\ \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1\ \ y_1$ and $\ y_2 \in C$.

Using the parallelogram equality we obtian

$$\begin{aligned} &\parallel \alpha_1 y_1 + \alpha_2 y_2 - tx \parallel^2 \\ &\qquad = \alpha_1^2 \parallel y_1 - tx \parallel^2 + \alpha_2^2 \parallel y_2 - tx \parallel^2 + 2\alpha_1\alpha_2 < y_1 - tx, y_2 - tx >, \qquad (*) \end{aligned}$$

and

$$2 < y_1 - tx, y_2 - tx > = - \parallel y_1 - y_2 \parallel^2 + \parallel y_1 - tx \parallel^2 + \parallel y_2 - tx \parallel^2 . \qquad (**)$$

From $(*)$ and $(**)$ we obtain

$$\| \alpha_1 y_1 + \alpha_2 y_2 - tx \|^2 + \alpha_1\alpha_2 \| y_1 - y_2 \|^2 \\ \leq \alpha_1 \| y_1 - tx \|^2 + \alpha_2 \| y_2 - tx \|^2 .$$

Taking the limitsup in the above legality as $t \to \infty$ we obtain (2). Setting, $u = \phi(x)$ and $\lambda(x) = v_x(u)$, we take $y_1 = \phi(x)$, and $y_2 = y \in C$ in (2) we obtain

$$\lambda^2(x) + \alpha_1\alpha_2 \| \phi(x) - y \|^2 \leq v_x^2(\alpha_1\phi(x) + \alpha_2 y) + \alpha_1\alpha_2 \| \phi(x) - y \|^2 \\ \leq \alpha_1\lambda^2(x) + \alpha_2 v_x^2(y).$$

So

$$\lambda^2(x) + \alpha_1 \| \phi(x) - y \|^2 \leq v_x^2(y)\,, \quad \forall\ \alpha_1 \in [0,1].$$

Consequently we obtain

$$\lambda^2(x) + \| \phi(x) - y \|^2 \leq v_x^2(y). \tag{3}$$

On the other hand we have

$$\begin{aligned} v_x(ty) &= \overline{\lim}_{s\in S} \| ty - sx \| \\ &= \overline{\lim}_{s\in S} \| ty - tsx \| \quad \text{from Lemma 2} \\ &\leq a(t)\overline{\lim}_{s\in S} \| sx - y \| + b(t)(\overline{\lim}_{s\in S} \| tsx - y \| \\ &\quad + \overline{\lim}_{s\in S} \| ty - sx \|) \\ &\leq (a(t) + b(t))v_x(y) + b(t)v_x(ty). \end{aligned}$$

Moreover,

$$v_x(ty) \leq (\frac{a(t) + b(t)}{1 - b(t)})v_x(y).$$

Therefore, replacing in (3) y by $t\phi x$, we obtain

$$\lambda^2(x) + \| \phi(x) - t\phi x \|^2 \leq (\frac{a(t) + b(t)}{1 - b(t)})^2\lambda^2(x). \tag{4}$$

Taking the limitsup in the above legality as $t \to \infty$, we get

$$v_{\phi x}^2(\phi x) \leq K\lambda^2(x) \;\; \text{with} \;\; K = \overline{\lim_t}(\frac{a(t) + b(t)}{1 - b(t)})^2 - 1 < 1$$

from (1).

Using (3), we obtain the following legality

$$\lambda^2(\phi x) \leq \lambda^2(\phi x) + \parallel \phi(x) - \phi^2 x \parallel^2 \leq v^2_{\phi x}(\phi x) \leq K\lambda^2(x)\,.$$

This implies that

$$\lambda^2(\phi^n x) \leq K^n \lambda^2(x).$$

From the legality (3) we have

$$\parallel \phi^{n+1}(x) - \phi^n(x) \parallel^2 \leq K^n \lambda^2(x).$$

Then the sequence $\{\phi^n(x)\}$ strongly converge since $\sum_{i=0}^{n-1} ||(\phi^{n+1}(x) - \phi^n(x))||$ converge. Let f is the limit of $\{\phi^n(x)\}$. For $t \in S$ from (4), we have

$$||\phi^n(x) - t\phi^n(x)|| \leq (k_t^2 - 1)K^{n-1}\lambda(x).$$

Letting the limit as $n \to \infty$, we obtain

$$\parallel f - tf \parallel = 0.$$

Concluding that the set of fixed point of S is nonempty. □

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