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WEAKLY g-INVERTIBLE OPERATORS IN BANACH SPACES AND THEIR MOORE-PENROSE INVERSE

S. Lahrech[1 §], A. Ouahab[2], A. Benbrik[3], A. Mbarki[4]

[1,2,3,4]Department of Mathematics
Faculty of Science
Mohamed First University
Oujda, MOROCCO

[1]e-mail: lahrech@sciences.univ-oujda.ac.ma
[2]e-mail: ouahab@sciences.univ-oujda.ac.ma
[3]e-mail: benbrik@sciences.univ-oujda.ac.ma
[4]e-mail: mbarki@sciences.univ-oujda.ac.ma

Abstract: In this paper, we define a new class of linear operators called class of weakly g-invertible operators which contains the usual class of g-invertible operators. For such operators, we define their generalized inverse according to a fixed algebrical decomposition. Finally, we prove the existence and the unicity of the Moore-Penrose inverse of weakly g-invertible operators. Our result generalize the result established in [3] for g-invertible operators.

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1. Weakly g-Invertible Operators

1.1. g-Invertible Operators

Let A be a closed operator on a Banach space X. Let $\mathcal{D}(A)$, $R(A)$ and $N(A)$

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[§]Correspondence author

denote respectively, the domain of A, the range of A and the null space of A.

Definition 1. The operator A is said to be g-invertible if there exists a bounded operator B from X into $\mathcal{D}(A)$ such that $ABA = A$ and $BAB = B$.

Remark 2. (see [1], [4]) 1) A is g-invertible if and only if $N(A)$ and $R(A)$ are closed in X and admit respectively a topological complement in X. In particular, if $X = H$ is a Hilbert space, then A is g-invertible if and only if $R(A)$ is closed in X.

2) Set $P = AB$ and $Q = BA$. Then P and Q are two bounded projections satisfying the conditions $R(P) = R(A)$ and $N(Q) = N(A)$.

3) If there exists a bounded operator B from X into $\mathcal{D}(A)$ such that $ABA = A$, then A is g-invertible and $S = BAB$ is a generalized inverse of A.

Definition 3. (see [3]) A is said to be topologically decomposable if the following conditions hold: $R(A)$ is closed in X, and there exists two closed subspaces E and F of X such that $X = N(A) \bigoplus E$ and $X = R(A) \bigoplus F$. In this case, we say that the pair (E, F) is a topological decomposition of A.

1.2. Weakly g-Invertible Operators

Let A be a linear operator (not necessarily closed) defined on a linear subspace $\mathcal{D}(A)$ of a Banach space $(X, \|.\|_X)$ with values in X. We will define now the class of weakly g-invertible operators which contains the usual class of g-invertible operators.

Definition 4. We say that A is weakly g-invertible (or algebrically decomposable) if there exists two subspaces E and F of X (not necessarily closed) such that $X = N(A) \bigoplus E$ and $X = R(A) \bigoplus F$. In this case, we say that the pair (E, F) is an algebrical decomposition of A or A is (E, F)-weakly g-invertible.

It is clear that every g-invertible operator A is weakly g-invertible.

1.3. Generalized Inverse of a Weakly g-Invertible Operator

Assume now that A is a weakly g-invertible operator. Let (E, F) be an algebrical decomposition of A. According to [2], there exists a norm $\|.\|_{R(A)}$ on $R(A)$ such that $(R(A), \|.\|_{R(A)})$ is continuously embedded in $(X, \|.\|_X)$. Let $\|.\|_1$ the norm defined on X by: $\|x\|_1 = \|x_1\|_{R(A)} + \|x_2\|_X$, where $x_1 \in R(A)$ and $x_2 \in F$. Such a decomposition is unique since algebrically $X = R(A) \bigoplus F$. So the norm $\|.\|_1$ is well defined.

Let us remark that the norm $\|.\|_1$ is finer than $\|.\|_X$ and $(X, \|.\|_1)$ is not necessarily a Banach space. But if A and F are closed in $(X, \|.\|_X)$, then $(X, \|.\|_1)$ becomes a Banach space (see [2]).

Definition 5. Let B be an operator acting from X into $\mathcal{D}(A)$. We say that B is a generalized inverse of A according to the algebrical decomposition (E, F) if the following conditions are satisfied:

1) B is closed from $(X, \|.\|_1)$ into $(\mathcal{D}(A), \|.\|_X)$;

2) $ABA = A$ and $BAB = B$.

Remark 6. 1) Even if A is a weakly g-invertible, $R(A)$ is not necessarily closed in $(X, \|.\|_X)$.

2) if A and F are closed in $(X, \|.\|_X)$, then the norms $\|.\|_X$ and $\|.\|_1$ are equivalent. Therefore, if B is a generalized inverse of A according to the decomposition (E, F), then B is a generalized inverse of A. Thus, A is g-invertible. This implies that $R(A)$ is closed in $(X, \|.\|_X)$ by ([1],[4]).

3) if A is g-invertible, then for every topological decomposition (E, F) of A the following conditions are equivalent:

i) B is a generalized inverse of A;

ii) B is a generalized inverse of A according to the decomposition (E, F).

Now we can prove the main result of our paper.

2. Existence and Unicity of the Moore-Penrose Inverse of Weakly g-Invertible Operators

Theorem 7. *Let A be a (E, F)-weakly g-invertible operator. Then there exists an unique generalized inverse of A according to the decomposition (E, F) satisfying the conditions: $R(BA) = E \bigcap \mathcal{D}(A)$ and $N(AB) = F$. We call it the Moore-Penrose inverse of A according to the decomposition (E, F).*

Proof. Let $\hat{A} = A \mid E \bigcap \mathcal{D}(A)$ the restriction of A to $E \bigcap \mathcal{D}(A)$. Then $\hat{A}$ is one-to-one and $R(\hat{A}) = R(A)$. Let B the operator defined from X into $\mathcal{D}(A)$ by: $B = \hat{A}^{-1}$ on $R(A)$ and $B = 0$ on F. Then B is closed from $(X, \|.\|_1)$ into $(X, \|.\|_X)$. Indeed, let $(x_n)_n$ a sequence of elements of X, x, y two elements of X such that $x_n \to x$ in $(X, \|.\|_1)$ and $Bx_n \to y$ in $(X, \|.\|_X)$. Set $y_n = Bx_n$. Since $X = R(A) \bigoplus F$, then $x_n = u'_n + v'_n$ with $u'_n \in R(A)$ and $v'_n \in F$. On the other hand, there exists an unique $u_n \in E \bigcap \mathcal{D}(A)$ such that $u'_n = \hat{A}u_n = Au_n$. Therefore, using the fact that $x_n \to x$ in $(X, \|.\|_1)$, we deduce that $u'_n \to Au$ in $(R(A), \|.\|_{R(A)})$. Hence, by the definition of $\|.\|_{R(A)}$ (see [2]) $Bu'_n \to B\hat{A}u = u$ in $(X, \|.\|_X)$. On the other hand, there exists an unique $u_n \in E \bigcap \mathcal{D}(A)$ such that $u'_n = \hat{A}u_n = Au_n$. It follows then that $u_n \to u$ in $(X, \|.\|_X)$. In this case,

$x_n = \hat{A}u_n + v'_n$. Thus, $y_n = Bx_n = B\hat{A}u_n + Bv'_n = u_n$. Therefore, $u_n \to y$ in $(X, \|.\|_X)$. Hence, $u = y$. Consequently, $Bx = B\hat{A}u + Bv' = u = y$. This implies that B is closed from $(X, \|.\|_1)$ into $(\mathcal{D}(A), \|.\|_X)$.

Let $x \in \mathcal{D}(A)$. Then there exists $x_1 \in E \cap \mathcal{D}(A)$ such that $Ax = Ax_1$. Therefore, $ABAx = AB\hat{A}x_1 = Ax_1 = Ax$. Hence, $ABA = A$.

Let now $x \in X = R(A) \oplus F$. Then $\exists!(u, v') \in (E \cap \mathcal{D}(A)) \times F$ such that $x = Au + v'$. Using the definition of B, we deduce that $BABx = BAu = BAu + Bv' = Bx$. This implies that B is a generalized inverse of A according to the decomposition (E, F).

Let us prove now that $R(BA) = E \bigcap \mathcal{D}(A)$ and $N(AB) = F$.

Let $x \in N(AB)$. Then $Bx \in N(A) \cap E = \{0\}$ which implies that $x \in N(B)$. Thus, $N(AB) = N(B)$. On the other hand, $N(B) = F$. Indeed, we have $F \subset N(B)$ and if $x \in N(B)$, then $\exists u \in E \cap \mathcal{D}(A)$ and $v' \in F$ such that $x = Au + v'$. Consequently, $u = Bx = 0$. Therefore, $x = v' \in F$. Hence, $N(AB) = N(B) = F$.

To prove that $R(BA) = E \cap \mathcal{D}(A)$ it suffices to remark that $R(BA) \subset R(B) \subset E \cap \mathcal{D}(A)$ and that if $x \in E \cap \mathcal{D}(A)$ then $BAx = B\hat{A}x = x$. So, we have proved that $R(BA) = E \bigcap \mathcal{D}(A)$ and $N(AB) = F$.

To conclude the proof, assume that there exists another generalized inverse B' of A according to the decomposition (E, F) and satisfying the conditions: $R(B'A) = E \bigcap \mathcal{D}(A)$ and $N(AB') = F$. Let $y \in R(A)$. Then there exists an unique $u \in \mathcal{D}(A) \cap E$ such that $y = \hat{A}u = Au$. Thus, $By = u$ and $B'y = B'Au \in R(B'A) = E \cap \mathcal{D}(A)$. This implies that $AB'y = AB'Au = Au = y$. Consequently, $u - B'y \in N(A) \cap E = \{0\}$. Hence, $B = B'$ on $R(A)$.

Let now $x \in F$. Since $F = N(AB') = R(I - AB')$, then $\exists u \in X$ such that $u - AB'u = x$. Therefore, $B'x = B'u - B'AB'u = 0$. Thus, $B = B'$ on F. □

References

[1] S.R. Caradus, Generalized inverses and operator theory, *Queen's Papers in Pure and Applied Mathematics*, Kingston, Ontario, **50** (1978).

[2] S. Lahrech, A. Ouahab, A. Mbarki, On the construction of operator range topology and topological uniform descent of Kato operators in Banach spaces, *International Journal of Pure and Applied Mathematics* (2005), Accepted.

[3] S. Lahrech, A. Ouahab, T.Ouahid. On the existence of the Moore-Penrose inverse of g-invertible operators in Banach spaces, *IJAM* (2005), Submitted.

[4] M.Z. Nashed, *Generalized Inverses and Applications*, Academic Press (1976).