International Journal of Pure and Applied Mathematics

Volume 23 No. 1 2005, 1-20

CONVEXITY AND UNIQUENESS OF THE SOLUTION TO THE LIOUVILLE EQUATION

Tomohiko Sato[1], Takashi Suzuki[2 §]

[1,2]Division of Mathematical Science
Department of System Innovation
Graduate School of Engineering Science
Osaka University
1-3 Machikaneyama, Toyonakashi, Osaka, 560-8531, JAPAN
[1]e-mail: t-sato@sigmath.es.osaka-u.ac.jp
[2]e-mail: suzuki@sigmath.es.osaka-u.ac.jp

Abstract: In this paper we study uniqueness and convexity of the level sets of the solution to the Liouville equation

$$-\Delta v = \lambda V(x) e^{v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $V(x) > 0$ is a positive-valued $C^1(\overline{\Omega})$ function, and $\lambda > 0$ is a constant.

AMS Subject Classification: 35J60
Key Words: Liouville equation, blow-up analysis, Green's function

1. Introduction

The purpose of the present paper is to study uniqueness and convexity of the level sets of the solution to the Liouville equation

$$-\Delta v = \lambda V(x) e^{v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary $\partial\Omega$, $V = V(x) > 0$ is a $C^1(\overline{\Omega})$ function, and $\lambda > 0$ is a constant. Putting

Received: May 10, 2005

§Correspondence author

$$\Lambda = \lambda \int_\Omega V(x) e^v,$$

we obtain the mean field equation

$$-\Delta v = \frac{\Lambda V(x) e^v}{\int_\Omega V(x) e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{2}$$

describing the equilibrium of the mean field of many vortices of perfect fluid in Onsagar's formulation [3, 4, 15], particularly, in the case of $V(x) \equiv 1$:

$$-\Delta v = \frac{\Lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{3}$$

In 1992, [26] showed that if Ω is simply-connected, then (2) admits a unique solution for each $\Lambda \in (0, 8\pi)$. Combining this with the previous work [29], [28] obtained a criterion for the existence of exactly two solutions for each $\lambda \in (0, \underline{\lambda})$ to

$$-\Delta v = \lambda e^v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where $\underline{\lambda}$ denotes the supremum of λ for the existence of the solution. More precisely, this case occurs if Ω is close to a disc, and the above mentioned criterion is given in terms of the conformal geometric quantity of Ω. This study is refined by [5], and particularly, it is proven that the solution to (3) for $\Lambda = 8\pi$ is at most one and the existence of the energy minimizing solution is characterized by a similar quantity.

However, such a criterion is not sufficient to characterize the domain Ω that admits exactly two solutions to (4) for $\lambda \in (0, \underline{\lambda})$. It is not sufficient also to characterize Ω that admits the formation of a one-dimensional manifold of the solution in $\lambda - v$ space, connecting the trivial solution $(\lambda, v) = (0, 0)$ and the one-point blow-up sigular limit $(\lambda, v) = (0, 8\pi G(\cdot, x_0))$, where $G(x, x')$ denotes the Green's function and $x_0 \in \Omega$ is a critical point of the Robin function $R = R(x)$ defined by

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}.$$

See [30, 21, 27, 1, 22] and the next section of this paper, concerning the singular limit of the solution.

For instance, although the convexity of Ω is an important factor for such a problem, if Ω is thin, then there is a solution to (3) with $\Lambda > 8\pi$ (see [20]). Thus, [26] is not applicable in this case, but still we have the asymptotic uniqueness of the solution. More precisely, the solution to (4) for $0 < \lambda \ll 1$ is exactly two, if Ω is convex and symmetric with respect to $x_1 = 0$ and $x_2 = 0$.

The reason is as follows. First, we have an a priori bound of $\Lambda = \lambda \int_\Omega e^v$ because Ω is simply connected [28], and therefore, only finite blow-up points are admitted as $\lambda \downarrow 0$. Next, since Ω is convex, there is only one critical point of the Robin function, denoted by x_0, and furthermore, it is non-degenerate [29, 20, 28]. Therefore, by [27] there is a unique portion of one-dimensional solution manifold in $\lambda - v$ space, denoted by $\overline{S} = \{(\lambda, \overline{v}_\lambda) \mid 0 < \lambda \ll 1\}$, such that $\overline{v}_\lambda \to 8\pi G(\cdot, x_0)$ locally uniformly in $\overline{\Omega} \setminus \{x_0\}$ as $\lambda \downarrow 0$. Finally, since Ω is convex and symmetric with respect to both axes, only the origin can be the blow-up point [10], i.e., otherwise we have infintely many blow-up points, a contradiction. Thus, we obtain the conclusion.

The argument [27] uses complex variables to guarantee the local uniqueness of one-point blow-up solution sequence, and is applicable to (1) for $V = |f'(z)|^2$, where $f = f(z)$ is a holomorphic function. Unfortunately, the other case is not within the scope of this argument, but recently, [12] obtained a real analytic proof for the above mentioned asymptotic uniqueness of the solution to (4) for convex and symmetric Ω. The employed argument is the blow-up analysis, and in the previous work [25], we extended it to $V(x) \not\equiv$ constant. Using this, now we can show the asymptotic uniqueness to (1) with $V(x) \not\equiv$ constant.

Theorem 1. *If Ω is symmetric and convex with respect to $x_1 = 0$ and $x_2 = 0$ and $V(x)$ is monotone decreasing in $x_1 > 0$ and $x_2 > 0$, then the solution to (1) with $0 < \lambda \ll 1$ is exactly two.*

Here and henceforth, we say that Ω is convex with respect to x_i $(i = 1, 2)$, if any segment parallel to the x_i axis with the endpoints in Ω is contained in Ω.

Similarly, we can extend a result of [12] to $V(x) \not\equiv$ constant concerning the uniqueness of the maximum point of the solution.

Theorem 2. *If $\Omega \subset \mathbb{R}^2$ is convex and $\{(\lambda_k, v_k)\}$ is a solution sequence to (1) blowing-up at one point, then we have $(x - x_k) \cdot \nabla v_k(x) < 0$ in $\Omega \setminus \{x_k\}$ for k sufficiently large, where $x_k \in \Omega$ denotes the maximum point of v_k, i.e., $v_k(x_k) = \max_{\overline{\Omega}} v_k$.*

In the next section, we describe the behavior of the non-compact solution sequence in more details. Non-variational construction of such a sequence is done for $V = |f'|^2$, where $f = f(z)$ is a holomorphic function by [27] and for the general V by [9] (see also [14]).

The final result is concerned with the convexity of the level sets of the solution, and extends [12] to $V(x) \not\equiv$ constant.

Theorem 3. *If $\partial\Omega$ has strictly positive curvature at any point and $\{(\lambda_k, v_k)\}$ is a solution sequence to (1) blowing-up at one point, then the level* *sets of v_k have strictly positive curvature at any point except for the maximum point x_k for k large, and therefore, these level sets are strictly convex.*

We note that the conclusions of the above two theorems are not involved by $V(x)$. This is because the singular limits of the solution to (1) and (4) are the same except for the location of singular points.

This paper is compose of five sections. Taking preliminaries in Section 2, Theorem 1, 2, 3 are proven in Sections 3, 4, 5, respectively.

2. Preliminaries

In this section, we review several results concerning (1). The first result is the classification of the singular limit [19] proven by the blow-up analysis. Here, the case $V(x) \equiv 1$ was obtained by [22], using complex variables.

Theorem 4. *If $\{(\lambda_k, v_k)\}$ is a solution sequence to (1) satisfying $\lambda_k \downarrow 0$, then we have a subsequence, denoted by the same symbol, such that $\Lambda_k = \lambda_k \int_\Omega V_k(x) e^{v_k} \to 8\pi m$ for some $m = 0, 1, 2, \ldots, +\infty$. According to this value of m, we have the following:*

1. *If $m = 0$, then it holds that $\|v_k\|_\infty \to 0$.*
2. *If $m \in \mathbb{N}$, then the blow-up sets of $\{v_k\}$, defined by*

$$\mathcal{S} = \left\{ x_0 \in \overline{\Omega} \mid \text{there exists } x_k \to x_0 \text{ such that } v_k(x_k) \to +\infty \right\},$$

is composed of m-interior points, and $v_k \to 8\pi \sum_{x_0 \in \mathcal{S}} G(\cdot, x_0)$ locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$, where $G = G(x, x')$ denotes the Green's function of $-\Delta$ in Ω with $\cdot|_{\partial\Omega} = 0$. We have $-\Delta v(x) dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} 8\pi \delta_{x_0}(dx)$ in the sense of measures on $\overline{\Omega}$ and furthermore, it holds that

$$\frac{1}{2} \nabla R(x_0) + \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} \nabla_x G(x_0, x_0') + \frac{1}{8\pi} \nabla \log V(x_0) = 0 \tag{5}$$

for each $x_0 \in \mathcal{S}$, where $R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}$ is the Robin function.

3. *If $m = +\infty$, then $v_k \to +\infty$ locally uniformly in Ω.*

Especially when $m = 1$, the blow-up point x_0 is a critical point of $R(x) + \frac{1}{4\pi} \log V(x)$. The following result guarantees that its non-degeneracy implies that of the linearized operator [25], and therefore, it is the extension of [12] to $V(x) \not\equiv \text{constant}$.

Theorem 5. *If $x = x_0 \in \Omega$ is a non-degenerate critical point of*

$$R(x) + \frac{1}{4\pi}\log V(x),$$

$V(x)$ is C^2 around $x = x_0$, and $\{(\lambda_k, v_k)\}$ is a solution sequence to (1) blowing-up at x_0, then the linearized operator $-\Delta - \lambda_k V(x)e^{v_k}$ in Ω with $\cdot|_{\partial\Omega} = 0$ is invertible.

Finally, we recall the following result valid to more general problems [10].

Theorem 6. *If $\Omega \subset \mathbb{R}^2$ is convex and symmetric with respect to $x_1 = 0$ and $x_2 = 0$, $V(x)$ is symmetric with respect to $x_1 = 0$ and $x_2 = 0$ and monotone decreasing for $x_1 > 0$ and $x_2 > 0$, and (λ, v) is a solution to (1), then $v = v(x)$ is symmmetric with respect to $x_1 = 0$ and $x_2 = 0$, and it holds that $\frac{\partial v}{\partial x_i} < 0$ for $x_i > 0$ $(i = 1, 2)$. Thus, only the origin is the maximum point of v.*

3. Proof of Theorem 1

First, we confirm the following lemma, known more or less, as is described in the introduction.

Lemma 1. *Under the assumptions of Theorem 1, if $\{(\lambda_k, v_k)\}$ is a solution sequence to (1) satisfying $\lambda_k \downarrow 0$, then it holds that*

$$\lim_{k\to\infty} \lambda_k \int_\Omega V e^{v_k} = 0 \qquad \text{or} \qquad \lim_{k\to\infty} \lambda_k \int_\Omega V e^{v_k} = 8\pi.$$

Proof. From the Pohozaev identity [24], we obtain

$$2\lambda_k \int_\Omega V(x)\left(e^{v_k} - 1\right) = \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial v_k}{\partial \nu_x}\right)^2 (x \cdot \nu),$$

while we have $x \cdot \nu \geq C_1 > 0$ on $\partial\Omega$ by the assumption, where $C_1 > 0$ is a constant. This implies

$$\begin{aligned}
\Lambda_k^2 &= \left(\int_\Omega \lambda_k V e^{v_k}\right)^2 = \left(\int_\Omega -\Delta v_k\right)^2 = \left(\int_{\partial\Omega} \frac{\partial v_k}{\partial \nu_x}\right)^2 \\
&\leq |\partial\Omega| \int_{\partial\Omega} \left(\frac{\partial v_k}{\partial \nu}\right)^2 \leq \frac{|\partial\Omega|}{C_1} \int_{\partial\Omega} \left(\frac{\partial v_k}{\partial \nu}\right)^2 (x \cdot \nu) \\
&= \frac{|\partial\Omega|}{C_1} 4\lambda_k \int_\Omega V(x)\left(e^{v_k} - 1\right) = \frac{4\,|\partial\Omega|}{C_1}\left\{\Lambda_k - \int_\Omega \lambda_k V\right\},
\end{aligned}$$

and therefore $\Lambda_k = O(1)$. The blouwp points are at most finite by Theorem 4, and hence the solution does not blow-up, or blows-up at the origin by Theorem 6. Then, we obtain the conclusion. □

Here, we mention that the Pohozaev identity, combined with the conformal mapping, guarantees $\Lambda_k = O(1)$ if Ω is simply-connected. See [28]. Now, we show the following lemma.

Lemma 2. *Under the assumptions of Theorem 1, the non-compact solution sequence $\{(\lambda_k, v_k)\}$ to (1) with $\lambda_k \downarrow 0$ is unique.*

Proof. If this is not the case, we have $\lambda_k \downarrow 0$ and the solutions $u_k \neq v_k$ to (1) with $\lambda = \lambda_k$, satisfying

$$\lambda_k \int_\Omega V(x)e^{u_k} \to 8\pi, \qquad \lambda_k \int_\Omega V(x)e^{v_k} \to 8\pi.$$

Both u_k and v_k take the maximum at the origin, and we define $\tilde{w}_k : \tilde{\Omega}_k \to \mathbb{R}$ by

$$\tilde{w}_k(x) = \frac{u_k(\delta_k x) - v_k(\delta_k x)}{\|u_k(\delta_k \cdot) - v_k(\delta_k \cdot)\|_{L^\infty(\tilde{\Omega}_k)}}, \quad \tilde{\Omega}_k = \left\{x \in \mathbb{R}^2 \mid \delta_k x \in \Omega\right\},$$

$$\delta_k^2 \lambda_k e^{\|u_k\|_\infty} = 1.$$

Then, we obtain

$$\begin{cases} -\Delta \tilde{w}_k = V(\delta_k x)\tilde{c}_k(x)\tilde{w}_k & \text{in } \tilde{\Omega}_k\,, \\ \tilde{w}_k = 0 \quad \text{on } \partial\tilde{\Omega}_k\,, \\ \|\tilde{w}_k\|_\infty = 1, \end{cases}$$

where $\tilde{c}_k(x) = \displaystyle\int_0^1 e^{tu_k(\delta_k x)+(1-t)v_k(\delta_k x)-\|u_k\|_\infty} dt$. Defining

$$\tilde{u}_k(x) = u_k(\delta_k x) - \|u_k\|_\infty\,, \quad \tilde{v}_k(x) = v_k(\delta_k x) - \|v_k\|_\infty\,.$$

On the other hand, we have

$$\left|\tilde{v}_k(x) + \log\left\{1 + \frac{1}{8}V(0)\left|x\right|^2\right\}^2\right| \le C_2\,,$$

$$\left|\tilde{u}_k(x) + \log\left\{1 + \frac{1}{8}V(0)\left|x\right|^2\right\}^2\right| \le C_2 \tag{6}$$

for any $x \in \tilde{\Omega}_k$ and $k = 1, 2, \cdots$, and also

$$\tilde{u}_k \to \tilde{u}_0 = \log \frac{1}{\left\{1 + \frac{1}{8}V(0)\,|x|^2\right\}^2}, \quad \tilde{v}_k \to \tilde{v}_0 = \log \frac{1}{\left\{1 + \frac{1}{8}V(0)\,|x|^2\right\}^2}$$

in $C^{2,\alpha}_{loc}(\mathbb{R}^2)$, where $C_2 > 0$ is a constant and $0 < \alpha < 1$ (see [8, 17, 12, 25]). This implies

$$\begin{aligned}
\tilde{c}_k(x) &= \int_0^1 e^{tu_k(\delta_k x)+(1-t)v_k(\delta_k x)-\|u_k\|_\infty} dt \\
&= \int_0^1 e^{t\tilde{u}_k(x)+(1-t)\tilde{v}_k(x)+(1-t)\left(\|v_k\|_\infty - \|u_k\|_\infty\right)} dt \\
&\to \int_0^1 e^{t\tilde{u}_0(x)+(1-t)\tilde{v}_0(x)} dt = \frac{1}{\left\{1 + \frac{1}{8}V(0)\,|x|^2\right\}^2} \qquad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}^2).
\end{aligned}$$

Therefore, from the standard elliptic regularity [11] applied to $\tilde{w}_k = \tilde{w}_k(x)$, we obtain a subsequence, denoted by the same symbol, and $\tilde{w}_0 = \tilde{w}_0(x)$ such that

$$\begin{aligned}
&\tilde{w}_k \to \tilde{w}_0 \qquad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}^2), \\
&-\Delta \tilde{w}_0 = \frac{V(0)}{\{1 + \frac{1}{8}V(0)\,|x|^2\}^2}\tilde{w}_0 \qquad \text{in } \mathbb{R}^2, \\
&\|\tilde{w}_0\|_\infty \le 1.
\end{aligned}$$

This implies

$$\tilde{w}_0(x) = \sum_{i=1}^{2} \frac{(a_i x_i)/\sqrt{c}}{\frac{8}{c} + |x|^2} + b \cdot \frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2} \qquad \text{in } \mathbb{R}^2$$

by [12], where $a_i, b \in \mathbb{R}$ and $c = V(0)$. Since $\tilde{w}_k = \tilde{w}_k(x)$ is symmetric with respect to $x_1 = 0$ and $x_2 = 0$, so is $\tilde{w}_0$. Then, $a_i = 0$ $(i = 1, 2)$ holds true:

$$\tilde{w}_0(x) = b \cdot \frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2}.$$

Here,

$$w_k(x) = \frac{u_k(x) - v_k(x)}{\|u_k - v_k\|_{L^\infty(\Omega)}},$$

satisfies

$$-\Delta w_k = V(x)c_k(x)w_k(x) \quad \text{in } \Omega, \qquad w_k = 0 \quad \text{in } \partial\Omega,$$

where $c_k(x) = \lambda_k \int_0^1 e^{tu_k(x)+(1-t)v_k(x)}dt$. Since

$$-u_k\Delta w_k = V(x)c_k(x)u_k w_k\,, \quad -w_k\Delta u_k = \lambda_k V(x)e^{u_k}w_k\,,$$

we obtain

$$0 = \int_{\partial\Omega}\left(u_k\frac{\partial w_k}{\partial\nu} - w_k\frac{\partial u_k}{\partial\nu}\right) = \int_\Omega V(x)\left(-c_k(x)u_k w_k + \lambda_k e^{u_k}w_k\right)$$

by Green's formula, and therefore,

$$\lambda_k\int_\Omega V(x)e^{u_k}w_k = \int_\Omega V(x)c_k(x)u_k w_k$$

follows. This implies

$$\begin{aligned}\lambda_k\int_\Omega V(x)e^{u_k}w_k &= \int_\Omega V(x)c_k(x)w_k(x)\left(u_k(x) - \|u_k\|_\infty\right)dx\\ &\quad + \|u_k\|_\infty\int_\Omega V(x)c_k(x)w_k(x)dx\\ &= \int_{\tilde\Omega_k}\tilde V_k(x)c_k(\delta_k x)w_k(\delta_k x)\left(u_k(\delta_k x) - \|u_k\|_\infty\right)\delta_k^2 dx\\ &\quad + \|u_k\|_\infty\int_{\tilde\Omega_k}\tilde V_k(x)c_k(\delta_k x)w_k(\delta_k x)\delta_k^2 dx\\ &= \int_{\tilde\Omega_k}\tilde V_k(x)\tilde c_k(x)\tilde w_k(x)\tilde u_k(x)dx + \|u_k\|_\infty\int_{\tilde\Omega_k}\tilde V_k(x)\tilde c_k(x)\tilde w_k(x)dx\end{aligned}$$

by $\delta_k^2 c_k(\delta_k x) = \tilde c_k(x)$, where $\tilde V_k(x) = V(\delta_k x)$. Here, from the Dominated Convergence Theorem based on the estimate [17], we have

$$\begin{aligned}&\int_{\tilde\Omega_k}\tilde V_k(x)\tilde c_k(x)\tilde w_k(x)\tilde u_k(x)dx\\ &\to \int_{\mathbb{R}^2}\frac{c}{\left\{1+\frac{c}{8}|x|^2\right\}^2}\cdot b\cdot\frac{\frac{8}{c}-|x|^2}{\frac{8}{c}+|x|^2}\cdot\log\frac{1}{\left\{1+\frac{c}{8}|x|^2\right\}^2} = 8\pi b,\end{aligned}$$

and hence it holds that

$$\lambda_k\int_\Omega V(x)e^{u_k}w_k = 8\pi b + \|u_k\|_\infty\int_{\tilde\Omega_k}\tilde V_k(x)\tilde c_k(x)\tilde w_k(x)dx + o(1).$$

We have, similarly,

$$
\begin{aligned}
&\lambda_k \int_\Omega V(x)e^{u_k} w_k = \lambda_k \int_{\tilde{\Omega}_k} V(\delta_k x)e^{u_k(\delta_k x)} w_k(\delta_k x)\delta_k^2 dx \\
&= \lambda_k \int_{\tilde{\Omega}_k} V(\delta_k x)e^{u_k(\delta_k x)} \tilde{w}_k(x)\lambda_k^{-1} e^{-\|u_k\|_\infty} dx \\
&= \int_{\tilde{\Omega}_k} V(\delta_k x)e^{\tilde{u}_k(x)} \tilde{w}_k(x) dx \\
&= \int_{\mathbb{R}^2} \frac{c}{\left\{1+\frac{1}{8}c\,|x|^2\right\}^2} \cdot b \cdot \frac{\frac{8}{c}-|x|^2}{\frac{8}{c}+|x|^2} dx + o(1) = o(1).
\end{aligned}
$$

Now, we shall show

$$
\|u_k\|_\infty \int_{\tilde{\Omega}_k} \tilde{V}_k(x)\tilde{c}_k(x)\tilde{w}_k(x)dx = o(1), \tag{7}
$$

and hence $b = 0$. In fact, by the divergence theorem, it holds that

$$
-\int_{\partial\Omega} \frac{\partial w_k}{\partial \nu} = \int_\Omega V(x)c_k(x)w_k(x)dx,
$$

and here, we can show

$$
\frac{\partial w_k}{\partial x_i} = o(\delta_k) \qquad (i = 1, 2). \tag{8}
$$

This implies (7) by

$$
\begin{aligned}
&\|u_k\|_\infty \int_\Omega V(x)c_k(x)w_k(x)dx = \|u_k\|_\infty \cdot o(\delta_k) \\
&= o\left(\delta_k \|u_k\|_\infty\right) = o\left(\delta_k \log \lambda_k\right) = o(1).
\end{aligned}
$$

To show (8), we use Green's representation formula, i.e.,

$$
\begin{aligned}
\frac{\partial w_k}{\partial x_i} &= \int_\Omega -\frac{\partial G(x,y)}{\partial x_i} \Delta w_k(y)dy \\
&= \int_\Omega -\frac{\partial G(x,y)}{\partial x_i} V(y)c_k(y)w_k(y)dy \\
&= \int_{\tilde{\Omega}_k} \frac{\partial G(x,\delta_k y)}{\partial x_i} \tilde{V}_k(y)\tilde{c}_k(y)\tilde{w}_k(y)dy. \qquad (9)
\end{aligned}
$$

Defining

$$f_k(y) = \tilde{V}_k(y)\tilde{c}_k(y)\tilde{w}_k(y)\,,$$
$$\xi_k(y) = \log\left[\frac{1}{|y|^2}\int_{-\infty}^{|y|} y f_k\left(t\frac{y_1}{|y|}, t\frac{y_2}{|y|}\right)dt\right],$$

we have

$$f_k(y) \to \frac{c}{\left\{1+\frac{1}{8}c\,|y|^2\right\}^2}\cdot b\cdot\frac{\frac{8}{c}-|y|^2}{\frac{8}{c}+|y|^2}$$

locally uniformly in y and

$$\left(y_1\frac{\partial\xi_k}{\partial y_1}+y_2\frac{\partial\xi_k}{\partial y_2}+2\right)e^{\xi_k(y)} = f_k(y),$$

and therefore,

$$\begin{aligned}
\frac{\partial w_k(x)}{\partial x_i} &= \int_{\tilde{\Omega}_k}\frac{\partial G(x,\delta_k y)}{\partial x_i}\left\{y_1\frac{\partial\xi_k}{\partial y_1}+y_2\frac{\partial\xi_k}{\partial y_2}+2\right\}e^{\xi_k(y)}dy \\
&= \sum_{j=1}^{2}\int_{\tilde{\Omega}_k}\frac{\partial G(x,\delta_k y)}{\partial x_i}\frac{\partial}{\partial y_j}\left\{y_j e^{\xi_k(y)}\right\}dy \\
&= -\delta_k\sum_{j=1}^{2}\int_{\tilde{\Omega}_k}\frac{\partial^2 G(x,\delta_k y')}{\partial x_i\partial y_j}y_j' e^{\xi_k(y')}dy' \\
&= \delta_k\left[\sum_{j=1}^{2}\frac{\partial^2 G(x,0)}{\partial x_i\partial y_j}\int_{\mathbb{R}^2} y_j e^{\frac{b}{2}\frac{64}{(8+c|y|^2)^2}}dy + o(1)\right] \\
&= o(\delta_k)
\end{aligned}$$

locally uniformly in $x\in\overline{\Omega}\setminus\{0\}$. This implies (8), and hence (7).

Since $b=0$ is proven, we obtain $\tilde{w}_0\equiv 0$ in $\mathbb{R}^2$. Therefore, if $\tilde{x}_k\in\tilde{\Omega}_k$ denotes the maximum point of $\tilde{w}_k$,

$$\tilde{w}_k(\tilde{x}_k) = \max_{\tilde{\Omega}_k}\tilde{w}_k = 1,$$

then it holds that $|\tilde{x}_k|\to\infty$ by $\tilde{w}_k\to\tilde{w}_0\equiv 0$ in $C^{2,\alpha}_{loc}(\mathbb{R}^2)$. Using the Kelvin transformation

$$\hat{u}_k(x) = \tilde{u}_k\left(\frac{x}{|x|^2}\right),\quad \hat{v}_k(x) = \tilde{v}_k\left(\frac{x}{|x|^2}\right),\quad \hat{w}_k(x) = \tilde{w}_k\left(\frac{x}{|x|^2}\right),$$

we have

$$\hat{w}_k\left(\frac{\tilde{x}_k}{|\tilde{x}_k|^2}\right)=\tilde{w}_k\left(\frac{|\tilde{x}_k|^4}{|\tilde{x}_k|^2}\frac{\tilde{x}_k}{|\tilde{x}_k|^2}\right)=1\,, \tag{10}$$

$$-\Delta\hat{w}_k(x)=\frac{1}{|x|^4}\tilde{V}_k\left(\frac{x}{|x|^2}\right)\hat{c}_k(x)\hat{w}_k(x)\qquad \text{in } \mathbb{R}^2\setminus\{0\} \tag{11}$$

in the sense of distributions, where 0 extensions of $\tilde{u}_k$, $\tilde{v}_k$ are taken where they are not defined, and $\hat{c}_k(x)=\tilde{c}_k\left(x/\left|x\right|^2\right)$. The right-hand side of (11), on the other hand, is uniformly bounded by (6), and therefore, $x=0$ is a removable singularity of $\hat{w}_k$. By the interior elliptic estimate, there exists a constant $C>0$ such that

$$\left\|\hat{w}_k\right\|_{L^\infty(B_{1/2}(0))}\le C\left\|\hat{w}_k\right\|_{L^2(B_1(0))}\,,$$

and the right-hand side converges to 0 by $\hat{w}_k\to 0$ in $C^{2,\alpha}_{loc}(\mathbb{R}^2\backslash\{0\})$ and $|\hat{w}_k|\le 1$. This is impossible by (10) and the proof is complete. □

Now, we give the following proof.

Proof of Theorem 1. From [13] and the assumption to $V(x)$, $x_0=0$ is a non-degenerate critical point of $R(x)+\frac{1}{4\pi}\log V(x)$. Then, using Theorem 5 and Lemma 2, we can argue similarly to [12] concerning the case of $V(x)\equiv$ constant. Then, we obtain the result. □

4. Proof of Theorem 2

If this is not the case, there is $y_k\in\Omega$ satisfying $y_k\neq x_k$ and

$$(y_k-x_k)\cdot\nabla v_k(y_k)\ge 0. \tag{12}$$

We may assume $y_k\to y_0\in\overline{\Omega}$. Defining $\delta_k^2=\lambda_k^{-1}e^{-\|v_k\|_\infty}$, we have the following cases:

Case 1. $y_0\neq x_0$.

Case 2. $y_0=x_0$ and $y_k\in B_{\delta_k R}(x_k)$ for some $R>0$.

Case 3. $y_0=x_0$, while $y_k\notin B_{\delta_k R}(x_k)$ for any $R>0$.

In Case 1, we have $\delta>0$ such that $y_0\in\overline{\Omega}\backslash B_\delta(x_0)$. Since $\Lambda_k=\lambda_k\int_\Omega V(x)e^{v_k}\to 8\pi$, we have

$$v_k\to 8\pi G(\cdot,x_0)\qquad \text{in } C^{2,\alpha}_{loc}(\overline{\Omega}\setminus\{x_0\})\,,$$
$$\nabla v_k\to 8\pi\nabla_x G(\cdot,x_0)\qquad \text{in } C^{1,\alpha}_{loc}(\overline{\Omega}\setminus\{x_0\}),$$

and therefore,

$$(y_k - x_k) \cdot \nabla v_k(y_k) \to 8\pi(y_0 - x_0) \cdot \nabla_x G(y_0, x_0) \geq 0. \tag{13}$$

We shall show that (13) is impossible.

In fact, $h(x) = (x - x_0) \cdot \nabla_x G(x, x_0)$ is harmonic in $\Omega \setminus B_\delta(x_0)$, and it holds that

$$h(x) = (x - x_0) \cdot \nabla G(x, x_0) < 0 \tag{14}$$

on $\partial\Omega$ by the Hopf Lemma and the convexity of Ω. On the other hand, $K(x, x') = G(x, x') + \frac{1}{2\pi} \log |x - x'|$ is smooth in $\Omega \times \Omega$, and we have

$$(x - x_0) \cdot \nabla_x G(x, x_0) = -\frac{1}{2\pi} + (x - x_0) \cdot \nabla_x K(x, x_0).$$

For $0 < \delta \ll 1$ we obtain

$$(x - x_0) \cdot \nabla_x G(x, x_0) < 0 \tag{15}$$

on $\partial B_\delta(x_0)$, and therefore, the maximum principle guarantees

$$(x - x_0) \cdot \nabla_x G(x, x_0) < 0 \tag{16}$$

in $\Omega \setminus B_\delta(x_0)$. Thus, (13) is impossible.

In Case 2, of $y_0 = x_0$ and $y_k \in B_{\delta_k R}(x_k)$ for $R > 0$, we define

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) - \|v_k\|_{L^\infty(\Omega)}\ , \ \tilde{\Omega}_k = \delta_k^{-1} (\Omega - \{x_k\})\ , \ \tilde{y}_k = (y_k - x_k)/\delta_k.$$

Since $|\tilde{y}_k| = \delta_k^{-1} |y_k - x_k| < R$, we may assume $\tilde{y}_k \to \tilde{y}_0 \in \overline{B_R(0)}$. First, we have

$$\tilde{y}_k \cdot \nabla \tilde{v}_k(\tilde{y}_k) = \frac{1}{\delta_k}(y_k - x_k) \cdot \nabla_x v_k(\delta_k x + x_k)|_{x=\tilde{y}_k} = (y_k - x_k) \cdot \nabla v_k(x_k) \geq 0\,, \tag{17}$$

by (12). We have also

$$\tilde{v}_k(x) \to \tilde{v}_0(x) = \log \frac{1}{\left\{1 + \frac{1}{8} V(x_0)\, |x|^2\right\}^2} \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^2)$$

by [18], and therefore,

$$\tilde{y}_k \cdot \nabla \tilde{v}_k(\tilde{y}_k) \to \tilde{y}_0 \cdot \nabla \tilde{v}_0(\tilde{y}_0) = -\frac{1}{2} \cdot \frac{V(x_0)\, |\tilde{y}_0|^2}{1 + \frac{1}{8} V(x_0)\, |\tilde{y}_0|^2}.$$

If $\tilde{y}_0 \neq 0$, then $\tilde{y}_0 \cdot \nabla \tilde{v}_0(\tilde{y}_0) < 0$. This is impossible by (17), and hence $\tilde{y}_0 = 0$ follows.

Defining $\phi_k(t) = \tilde{v}_k(t\tilde{y}_k)$, we obtain

$$\phi_k(t) = v_k(t(y_k - x_k) + x_k) - \|v_k\|_\infty \leq 0$$

for any t. Since $\phi_k(0) = 0$, there exists $s_k \in (0,1)$ such that

$$\phi_k'(s_k) = \frac{\phi_k(1) - \phi_k(0)}{1 - 0} = v_k(y_k) - \|v_k\|_\infty \leq 0.$$

We have also

$$\phi_k'(1) = (y_k - x_k) \cdot \nabla v_k(y_k) \geq 0$$

and therefore, there exists $r_k \in [s_k, 1]$ such that $\phi_k'(r_k) = 0$. We have, furthermore,

$$\phi_k'(0) = (y_k - x_k) \cdot \nabla v_k(x_k) = 0,$$

and hence $\bar{t}_k \in (0, r_k) \subset (0,1)$ such that

$$\phi_k''(\bar{t}_k) = \frac{d^2}{dt^2}\tilde{v}_k(\bar{t}_k\tilde{y}_k) = \frac{\phi_k'(r_k) - \phi_k'(0)}{r_k - 0} = 0.$$

This implies

$$\frac{d^2}{dt^2}\tilde{v}_0(0) = 0$$

by $\tilde{y}_0 = 0$ and therefore, $x = 0$ is a degenerate critical point of $\tilde{v}_0 = \tilde{v}_0(x)$.

Actually, $\nabla \tilde{v}_0(x) = -\frac{1}{2} \cdot \frac{V(x_0)x}{1 + \frac{1}{8}V(x_0)|x|^2}$ and $x = 0$ is a critical point of $\tilde{v}_0$, but the Hesse matrix of $\tilde{v}_0$ at 0 is

$$\begin{pmatrix} -\frac{V(x_0)}{2} & 0 \\ 0 & -\frac{V(x_0)}{2} \end{pmatrix}.$$

This is invertible, and we obtain a contradiction.

In the third case of $y_0 = x_0$ and $y_k \notin B_{\delta_k R}(x_k)$ for any $R > 0$, we have

$$r_k \equiv |y_k - x_k| \to 0 \qquad \text{and} \qquad R\delta_k \leq r_k \leq 1. \tag{18}$$

Here, we define

$$\begin{aligned} w_k(x) &= v_k(r_k x + x_k) - \frac{1}{2\pi}\lambda_k \log \frac{1}{r_k} \int_\Omega V(y) e^{v_k(y)} dy\,, \\ \Omega_k &= r_k^{-1}\left(\Omega - \{x_k\}\right). \end{aligned}$$

From Green's representation formula we have

$$v_k(x) = \lambda_k \int_\Omega G(x,y)V(y)e^{v_k(y)}dy$$

and therefore,

$$w_k(x) = \lambda_k \int_\Omega G(r_k x + x_k, y')V(y')e^{v_k(y')}dy' - \frac{1}{2\pi}\lambda_k \log\frac{1}{r_k}\int_\Omega V(y')e^{v_k(y')}dy'$$

in Ω_k. Using $y' = \delta_k y + x_k$, we have

$$\begin{aligned}
w_k(x) &= \int_{\tilde{\Omega}_k} G(r_k x + x_k, \delta_k y + x_k)V(\delta_k y + x_k)e^{\tilde{v}_k(y)}dy \\
&\quad -\frac{1}{2\pi}\log\frac{1}{r_k}\int_{\tilde{\Omega}_k} V(\delta_k y + x_k)e^{\tilde{v}_k(y)}dy \\
&= \int_{\tilde{\Omega}_k}\left[\frac{1}{2\pi}\log\frac{1}{|r_k x - \delta_k y|} + K(r_k + x_k, \delta_k y + x_k)\right]V(\delta_k y + x_k)e^{\tilde{v}_k(y)}dy \\
&\quad -\frac{1}{2\pi}\log\frac{1}{r_k}\int_{\tilde{\Omega}_k} V(\delta_k y + x_k)e^{\tilde{v}_k(y)}dy \\
&= \frac{1}{2\pi}\int_{\tilde{\Omega}_k}\log\frac{1}{\left|x - \frac{\delta_k}{r_k}y\right|}V(\delta_k y + x_k)e^{\tilde{v}_k(y)}dy \\
&\quad +\int_{\tilde{\Omega}_k} K(r_k x + x_k, \delta_k y + x_k)V(\delta_k y + x_k)e^{\tilde{v}_k(y)}dy,
\end{aligned}$$

where $\tilde{\Omega}_k = \frac{\Omega - \{x_k\}}{\delta_k}$. For the second term, it holds that

$$\int_{\tilde{\Omega}_k} K(r_k x + x_k, \delta_k y + x_k)V(\delta_k y + x_k)e^{\tilde{v}_k(y)}dy \to 8\pi K(x_0, x_0)$$

by

$$\int_{\tilde{\Omega}_k} e^{\tilde{v}_k} \to \int_{\mathbb{R}^2}\frac{1}{\left\{1 + \frac{1}{8}V(x_0)\,|y|^2\right\}^2} = \frac{8\pi}{V(x_0)}.$$

Next, (18) implies

$$\frac{\delta_k}{r_k} < \frac{1}{R},$$

where $R > 0$ is arbitrary. Thus, it holds that

$$\frac{\delta_k}{r_k} \to 0.$$

From [17], we have

$$\left| v_k(y') - \log \frac{e^{v_k(x_k)}}{\left\{1 + \frac{1}{8}\lambda_k V(x_k) e^{v_k(x_k)} \left|y' - x_k\right|^2\right\}^2} \right| \leq C_3$$

for any $y' \in \overline{\Omega}$ and k, where $C_3 > 0$ is a constant (see [25]). Using $y' = \delta_k y + x_k$, we have

$$\left| \tilde{v}_k(y) + \log \left\{1 + \frac{1}{8} V(x_k) \left|y\right|^2\right\}^2 \right| \leq C_3$$

for $y \in \tilde{\Omega}_k$, and hence the dominated convergence theorem is applicable. Thus, we obtain

$$w_k(x) \to 4 \log \frac{1}{|x|} + 8\pi K(x_0, x_0) \tag{19}$$

locally uniformly in $\mathbb{R}^2 \setminus \{0\}$.

We can apply the same argument for $\nabla w_k(x)$, using

$$\begin{aligned} \nabla w_k(x) &= \lambda_k \int_{\Omega} \nabla_x \left[G(r_k x + x_k, y')\right] V(y') e^{v_k(y')} dy' \\ &= \int_{\tilde{\Omega}_k} \left[-\frac{1}{2\pi} \frac{r_k(r_k x - \delta_k y)}{\left|r_k x - \delta_k y\right|^2} + r_k (\nabla_x K)(r_k x + x_k, \delta y + x_k) \right] \\ &\quad \times V(\delta_k y + x_k) e^{\tilde{v}_k(y)} dy. \end{aligned}$$

Then, the dominated convergence theorem guarantees

$$\begin{aligned} &\int_{\tilde{\Omega}_k} r_k (\nabla_x K)(r_k x + x_k, \delta y + x_k) V(\delta_k y + x_k) e^{\tilde{v}_k(t)} dt \to 0\,, \\ &\int_{\tilde{\Omega}_k} -\frac{1}{2\pi} \frac{r_k(r_k x - \delta_k y)}{\left|r_k x - \delta_k y\right|^2} V(\delta_k y + x_k) e^{\tilde{v}_k(y)} dy \to -\frac{1}{2\pi} \frac{x}{|x|^2} 8\pi = -\frac{4x}{|x|^2}, \end{aligned}$$

and we obtain

$$\nabla w_k(x) \to -\frac{4x}{|x|^2}$$

locally uniformly in $\mathbb{R}^2 \setminus \{0\}$, i.e.,

$$w_k(x) \to 4 \log \frac{1}{|x|} + 8\pi K(x_0, x_0) \qquad \text{in } C^1_{loc}(\mathbb{R}^2 \setminus \{0\}). \tag{20}$$

Finally, we have $|\overline{y}_k| = 1$ for $\overline{y}_k = \frac{y_k - x_k}{r_k}$. Thus, we may assume $\overline{y}_k \to \overline{y}_0$ with $|\overline{y}_0| = 1$. Here, we have

$$\overline{y}_k \cdot \nabla w_k(\overline{y}_k) = (y_k - x_k) \cdot \nabla v_k(y_k) \geq 0$$

from the assumption, while (20) guarantees

$$\overline{y}_k \cdot \nabla w_k(\overline{y}_k) \to \overline{y}_0 \cdot -\frac{4\overline{y}_0}{|\overline{y}_0|^2} = -4.$$

This is a contradiction, and the proof is complete.

5. Proof of Theorem 3

First, each level set is a regular curve by Theorem 2. If the conclusion is false, then we have $\lambda_k \downarrow 0$ and $y_k \in \Omega$, such that $y_k \neq x_k$ and $K_{v_k}(y_k) \leq 0$ for $k = 1, 2, \cdots$, where $K_{v_k}(y_k)$ denotes the curvature of the level curve

$$\{y \in \Omega \mid v_k(y) = v_k(y_k)\}$$

at $y = y_k$. We have

$$K_{v_k}(y) = -\frac{1}{|\nabla v_k(y)|} \cdot \frac{{}^t z \cdot \mathrm{Hess}\,(v_k(y))\, z}{|z|^2}, \tag{21}$$

using the Hesse matrix $\mathrm{Hess}\,(v_k(y))$ of $v_k = v_k(y)$ at $y = y_k$, where $z \neq 0$ is a vector satisfying $z \cdot \nabla v_k(y) = 0$. Since the curvature on $\partial\Omega$ is positive, we may assume $y_k \to y_0 \in \Omega$. Similarly to the proof of Theorem 2, now we consider the following three cases, defining $\delta_k > 0$ by $\delta_k^2 \lambda_k e^{\|v_k\|_\infty} = 1$:

Case 1. $y_0 \neq x_0$.

Case 2. $y_0 = x_0$ and $y_k \in B_{\delta_k R}(x_k)$ for some $R > 0$.

Case 3. $y_0 = x_0$, while $y_k \notin B_{\delta_k R}(x_k)$ for any $R > 0$.

In the first case, we have $\delta > 0$ such that $y_0 \in \Omega \setminus B_\delta(x_0)$. Since $v_k \to 8\pi G(\cdot, x_0)$ in $C^{2,\alpha}_{loc}\left(\overline{\Omega} \setminus B_\delta(x_0)\right)$ by Theorem 4, we obtain

$$K_{G(y_0,x_0)}(y_0) \leq 0,$$

which, however, is a contradiction by [16].

In the second case of $y_0 = x_0$ and $y_k \in B_{\delta_k R}(x_k)$ for some $R > 0$, we define

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) - \|v_k\|_\infty\,, \quad \tilde{y}_k = \frac{y_k - x_k}{\delta_k}.$$

Then, using (21), we obtain

$$K_{\tilde{v}_k}(\tilde{y}_k) \quad = \quad -\frac{1}{|[\nabla \tilde{v}_k](\tilde{y}_k)|} \frac{{}^t x \cdot \mathrm{Hess}\,(\tilde{v}_k(\tilde{y}_k))\, x}{|x|^2}$$

$$= \quad -\frac{\delta_k^2}{\delta_k}\frac{1}{|\nabla v_k(y_k)|}\frac{{}^t x \cdot \mathrm{Hess}\,(v_k(y_k))\, x}{|x|^2} = \delta_k K_{v_k}(y_k) \le 0.$$

By $|\tilde{y}_k| = \left|\frac{y_k - x_k}{\delta_k}\right| \le R$, we may assume $\tilde{y}_k \to \tilde{y}_0 \in \overline{B_R(0)}$. If $\tilde{y}_0 \neq 0$, then we have $K_{\tilde{v}_0}(\tilde{y}_0) \le 0$. However, this is impossible because $\tilde{v}_0(x) = \log \frac{1}{\left\{1+\frac{1}{8}V(x_0)|x|^2\right\}^2}$ is radially symmetric. Thus, we have $\tilde{y}_0 = 0$.

Taking $\tau_k = \alpha\left(-\frac{\partial v_k}{\partial x_2}, \frac{\partial v_k}{\partial x_1}\right)(y_k) \neq 0$, we have

$$K_{\tilde{v}_k}(\tilde{y}_k) = -\frac{1}{|\nabla \tilde{v}_k(\tilde{y}_k)|}\frac{{}^t\tau_k \cdot \mathrm{Hess}\,(\tilde{v}_k(\tilde{y}_k))\,\tau_k}{|\tau_k|^2} \le 0,$$

and therefore,

$${}^t\tau_k \cdot \mathrm{Hess}\,(\tilde{v}_k(\tilde{y}_k))\,\tau_k \ge 0.$$

We can choose τ_k satisfying $|\tau_k| = 1$, and take a subsequence such that $\tau_k \to \tau_0$, where $|\tau_0| = 1$. This implies ${}^t\tau_0 \cdot \mathrm{Hess}\,(\tilde{v}_0(0))\,\tau_0 \ge 0$, which, however, is impossible because

$$\mathrm{Hess}\,(\tilde{v}_0(0)) = -\frac{V(x_0)}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is not positive semi-definite.

In the final case, we have $y_0 = x_0$ and $y_k \in B_{\delta_k R}(x_k)$ for any $R > 0$. Then, we define

$$\begin{aligned}
&r_k \equiv |y_k - x_k| \to 0\,,\\
&R\delta_k \le r_k \le 1\,,\\
&w_k(x) = v_k(r_k x + x_k) - \frac{1}{2\pi}\left(\log\frac{1}{r_k}\right)\lambda_k \int_\Omega V(y)e^{v_k(y)}dy\,,
\end{aligned}$$

similarly to the proof of Theorem 2. We have proven

$$w_k(x) \to x_0(x) = 4\log\frac{1}{|x|} + 8\pi K(x_0, x_0)$$

in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$, but this is valid in $C^2_{loc}(\mathbb{R}^2 \setminus \{0\})$ by the same argument.

We have $|\tilde{y}_k| = 1$ for $\tilde{y}_k = \frac{y_k - x_k}{r_k}$, and therefore, $\tilde{y}_k \to \tilde{y}_0$ with $|\tilde{y}_0| = 1$, taking a subsequence. By (21), on the other hand, we obtain

$$K_{w_k}(\tilde{y}_k) \quad = \quad -\frac{1}{|\nabla w_k(\tilde{y}_k)|}\frac{{}^t z \cdot \mathrm{Hess}\,(w_k(\tilde{y}_k))\, z}{|z|^2}$$

$$= \quad -\frac{r_k^2}{r_k}\left|\nabla v_k(y_k)\right| \frac{{}^t z \cdot \mathrm{Hess}\,(w_k(\tilde{y}_k))\, z}{|z|^2} = r_k K_{v_k}(y_k) \le 0\,,$$

for all k. This implies

$$K_{w_k}(\tilde{y}_k) \to K_{w_0}(\tilde{y}_0) \le 0,$$

a contradiction, because w_0 is radially symmetric. The proof is complete.

References

[1] S. Baraket, F. Pacard, Construction of singular limits for a semilinear elliptic equation in dimension 2, *Calc. Vari. in PDE*, **6** (1998), 1-38.

[2] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equations*, **16** (1991), 1223-1253.

[3] E. Caglioti, P.L. Lions, C. Marchioro, W. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, *Comm. Math. Phys.*, **143** (1992), 501-525.

[4] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II, *Comm. Math. Phys.*, **174** (1995), 229-260.

[5] S.-Y.-A. Chang, C.-C. Chen, C.-S. Lin, Extremal functions for mean field equation in two dimension, In: *Lectures on Partial Differential Equations: Proceedings in Honor of Luis Nirenberg's 75-th Birthday*, (Ed-s: S.-Y.A. Chang, C.-S. Lin, S.-T. Yau), International Press, New York (2003).

[6] C.-C. Chen, C.-S. Lin, On the symmetry of blow-up solutions to a mean field equation, *Ann. Inst. H. Poincaré, Analyse Non lineaire*, **18** (2001), 271-296.

[7] C.-C. Chen, C.-S. Lin, Sharp estimates for solutions to multibubbles in compact Riemann surfaces, *Comm. Pure Appl. Math.*, **55** (2002), 728-771.

[8] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.*, **63** (1991), 615-622.

[9] P. Esposito, M. Grossi, A. Pistoia, On the existence of blowing-up solutions for a mean field equation, Preprint.

[10] B. Gidas, W.-M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.*, **68** (1979), 209-243.

[11] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Second Edition, Springer, Berlin (1983).

[12] F. Gladiali, M. Grossi, Some results on the Gel'fand problem, *Comm. Partial Differential Equations*, **29** (2004), 1335-1364.

[13] M. Grossi, M., On the nondegeneracy of the critical points of the Robin function in symmetric domains, *Comptes Rend. Acad. Sci. Paris.*, **335** (2002), 1-4.

[14] A. Hamatake, T. Suzuki, Singular solutions for a nonlinear Dirichlet problem, *Comm. Appl. Anal.*, **4** (2000), 411-432.

[15] M.K.-H. Kiessling, Statistical mechanics of classical particles with logarithmic interactions, *Comm. Pure Appl. Math.*, **46** (1993), 27-56.

[16] J.L. Lewis, Capacitary function in convex rings, *Arch. Rat. Mech. Anal.*, **66** (1977), 201-224.

[17] Y.Y. Li, Harnack type inequality: the method of moving planes, *Comm. Math. Phys.*, **200** (1999), 421-444.

[18] Y.Y. Li, I. Shafrir, Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two, *Indiana Univ. Math. J.*, **43** (1994), 1255-1270.

[19] L. Ma, J.C. Wei, Convergence for a Liouville equation, *Comment. Math. Helv.*, **76** (2001), 506-514.

[20] N. Mizoguchi, T. Suzuki, Equations of gas combustion: S-shaped bifurcation and mushrooms, *J. Differential Equations*, **134** (1997), 183-215.

[21] J.L. Moseley, Asymptotic solutions for a Dirichlet problem with an exponential nonlinearity, *SIAM J. Math. Anal.*, **14** (1983), 719-735.

[22] K. Nagasaki, T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities, *Asymptotic Analysis*, **3** (1990), 173-188.

[23] H. Ohtsuka, T. Suzuki, Blow-up analysis for Liouville type equation in self-dual gauge field theories, *Commun. Contemp. Math.*, To Appear.

[24] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet. Math. Dokl.*, **6** (1965), 1408-1411.

[25] T. Sato, T. Suzuki, Asymptotic non-degeneracy of the solution to the Liouville-Gel'fand problem in two dimensions, Preprint.

[26] T. Suzuki, Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity, *Ann. Inst. Henri Poincaré, Analyse non linéaire*, **9** (1992), 367-398.

[27] T. Suzuki, Some remarks about singular perturbed solutions for Emden-Fowler equation with exponential nonlinearity, In: *Functional Analysis and Related Topics, Lecture Notes in Math.*, Springer, **1540** (1993).

[28] T. Suzuki, *Semilinear Elliptic Equations*, Gakkotosho, Tokyo (1994).

[29] T. Suzuki, K. Nagasaki, On the nonlinear eigenvalue problem $\Delta u + \lambda e^u = 0$, *Trans. Amer. Math. Soc.*, **39** (1988), 591-608.

[30] V.H. Weston, On the asymptotic solution of a partial differential equation with an exponential nonlinearity, *SIAM J. Math. Anal.*, **9** (1978), 1030-1053.