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RIGHT-SIDED IDEMPOTENT QUANTALES AND ORTHOMODULAR LATTICES

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To the Memory of Professor Víctor Neumann-Lara.

Abstract: Let Q a Gelfand quantale. If $R(Q)$ denotes the subquantale of Q of the right-sided elements, $R(Q)$ turns out to be an idempotent, right-sided quantale. The non-commutative binary operation & in this case is induced by a quantifier. Quantifiers for the lattice of the closed subspaces of a separable, infinite dimensional Hilbert space are trivial.

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1. Introduction

The purpose of this article is to begin a study of quantales and the relation with orthomodular latices. As is well known, orthomodular lattices have a very

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close relation with the algebraic foundations of quantum mechanics and with algebraic logic. Indeed, by the logic of (non-relativistic) quantum mechanics is thought of as being the lattice of closed subspaces of a separable infinite dimensional Hilbert space, see [2] for more details. Algebraic logic was introduced by P. Halmos in [6] for Boolean algebras and specially, the notion of a quantifier. M.F. Janowitz generalize this concept for orthomodular lattices, see [1] and [2] for a discussion of the logic of quantum mechanics.

Quantales was introduced in [8] as a possible algebraic foundation of quantum mechanics. We shall see what is the relation between orthomodular lattices and quantales; one of the main results of the present article are focused on a special kind of quantales: the idempotent, right-sided quantales. This is not surprising, many of the important examples are in fact idempotent, right-sided quantales or contain as a subquantale an idempotent,right-sided quantale, see the definition of a Gelfand quantale and the consequences of this kind of quantales.

In Section 1 we introduce the required concepts to study quantales and orthomodular lattices with special mention of the notion of a Gelfand quantale.

Section 2 talks about quantifers on an orthomodular lattice. We notice that the central cover of an element in a complete orthomodular lattice L induces a binary operation & making L an idempotent, right-sided quantale. An application of the previous results are given for the lattice of a separable, infinite dimensional Hilbert space.

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Section 1

We follow the definitions contained in the monograph written by K. Rosenthal (see [12] for more details).

Definition 1. A quantale Q is a complete lattice with bounds 0 and 1 together with an associative product & satisfying:

1. $a\&(\vee_{i\in I}b_i) = \vee_{i\in I}(a\&b_i)$;
2. $(\vee_{i\in I}a_i)\&b = \vee_{i\in I}(a_i\&b)$,

for all $a, b, a_i, b_i \in Q$ and any set I.

Moreover, we will say that an element a of the quantale Q is:

1. Right-sided if: $a\&1 \leq a$.
2. Left-sided if: $1\&a \leq a$.
3. Strictly right (left)-sided if: $a\&1 = a$ $(1\&a = a)$.
4. Two-sided if a is both (strictly) left and (strictly) right-sided.
5. Idempotent if: $a\&a = a$.

We shall say Q is an idempotent and right-sided quantale if any element a of Q is idempotent and right-sided. Similarly, we can define an idempotent and strict right-sided quantale. We shall consider the subsets $R(Q)$ and $L(Q)$ consisting of all right-sided elements and left-sided elements of Q, it is not hard to show that 0 and 1 belong to $R(Q)$ and it is closed under arbitrary suprema and &; i.e., $R(Q)$ is a subquantale of Q. The same claim can be proved for $L(Q)$. A morphism of quantales $f : Q \to P$ is a function preserving arbitrary suprema, & and preserves 1.

Example 2. If L is a complete lattice, the monoid of all sup-preserving endomorphisms of L is a quantale where & is the composition of functions. Also, any complete Heyting algebra is a quantale where & is equal to $\wedge$. As is well known, a complete Heyting algebra A is a complete lattice satisfying:

1. For any $a, b, c \in A$ $\; a \wedge b \leq c$ if and only if $a \leq b \to c$.

In other words, the binary operation $\wedge$ in a complete Heyting algebra has a right adjoint.

In [4], F.Borceux and G. Van Den Bossche proved that given any complete lattice $(Q, \leq)$ there is a one-to-one correspondence between binary operations $\& : Q \times Q \to Q$ making Q into an idempotent, right-sided quantale and closure operations $j : Q \to Q$ satisfying the following conditions:

1. $a \leq j(a)$.
2. $j(a \wedge j(b)) = j(a) \wedge j(b)$.
3. $a \wedge j(\vee_{i \in I} b_i) = \vee_{i \in I}(a \wedge j(b_i)$.
4. $(\vee_{i \in I} a_i) \wedge j(b) = \vee_{i \in I}(a_i \wedge j(b))$.

Moreover, $a\&b = a \wedge j(b)$. The function $j : Q \to Q$ is defined by $j(b) = 1\&b$ as the reader can check easily.

For the sake of completeness we shall introduce some definitions which allow us to consider *Gelfand quantales*.

Definition 3. We will say that the quantale Q is unital if there exists an element $e \in Q$ for wich $e\&a = a = a\&e$ for all $a \in Q$.

Definition 4. An involution is an unary operation $*: Q \to Q$ such that for all $a, b, a_i \in Q$,

1. $a^{**} = a$.
2. $(a\&b)^* = b^*\&a^*$.
3. $(\vee_{i\in I} a_i)^* = \vee_{i\in I} a_i^*$.

The quantale Q is called involutive if it has an involution.

Any complete Heyting algebra is an involutive quantale, the involution is the identity function. The simplest non-trivial example of an involutive quantale is given by taking an arbitrary set, let us say X and consider the relations of this set, we denote this quantale by $\mathrm{Rel}(X)$. The operation & is just the composition of relations and the $*$-operation is the opposite (or converse) of a given relation and the unit is the diagonal $\triangle_X$ of X.

If H is an arbitrary Hilbert space, consider the lattice $C(H)$ of all closed subspaces of H. We define $Q(H)$, an involutive quantale, by taking all the sup-preserving mappings of $C(H)$ to itself (see, [9] for details).

Definition 5. A quantale Q is a Gelfand quantale if Q is unital, involutive and for all $a \in R(Q)$ satisfies the following condition: $a\&a^*\&a = a$.

Since $a \in R(Q)$ if and only if $a^* \in L(Q)$ the above condition is equivalent to asking that $a\&a^*\&a = a$ for all $a \in L(Q)$.

Example 6. With the above notation $\mathrm{Rel}(X)$ and $Q(H)$ are Gelfand quantales. C.J. Mulvey and J. Pelletier study in detail these two examples, see [9] for more information. We shall see in Section 2 an alternative way of taking the example of the lattice of the relations of a set X.

Clearly, any complete Heyting algebra L is a Gelfand quantale.

Remark 7. If Q is an involutive quantale the subquantale $R(Q)$ of right-sided elements of Q can be endowed with a negation operation $\perp$ by defining $a^\perp = \vee_{a^*\&b=0} b$ for $a, b \in R(Q)$. That is, $a^\perp$ is the largest element of $R(Q)$ such that $a^*\&a^\perp = 0$; it is not hard to show also $a \wedge a^\perp = 0$.

Similarly, we have for $L(Q)$, the quantale of left-sided elements of Q, a negation operation $\perp$ defined by $b^\perp = \vee_{a\&b^*=0} a$ for $a, b \in L(Q)$; as above, $b^\perp$ is the largest element of $L(Q)$ such that $b^\perp\&b^* = 0$.

Moreover, the following is true:

1. For any $a \in R(Q)$: $a \leq (a^\perp)^\perp$.
2. For any family $\{a_i\}_{i\in I} \subset R(Q)$: $(\vee_{i\in I} a_i)^\perp = \wedge_{i\in I} a_i^\perp$.

We shall see: if Q is a Gelfand quantale then every element a of $R(Q)$ is in fact a strict right-sided and idempotent.

Lemma 8. *Let Q be a Gelfand quantale. Then $R(Q)$ the set of the right-sided elements of Q is an idempotent, strictly right-sided subquantale of Q.*

Proof. It is enough to prove that $a\&a = a$ and $a\&1 = a$, for all $a \in R(Q)$. We prove these two claims as follows:

$$a = a\&a^*\&a \leq a\&1\&a \leq a\&a \leq a\&1 \leq a.$$

Hence, $R(Q)$ is an idempotent and strictly right-sided quantale. □

Therefore, any Gelfand quantale Q induces in a natural way an idempotent and strictly right-sided quantale. Clearly, the same result holds for $L(Q)$. We know by [4] that for elements a, b in $R(Q)$, $a\&b = a \wedge (1\&b)$. Moreover, the map $F : R(Q) \to R(Q)$ given by $F(a) = 1\&a$ satisfies:

1. For any $a \in R(Q)$, $a \leq F(a)$.
2. F is idempotent.
3. For any $a, b \in R(Q)$, $F(a\&b) = F(a)\&F(b) = F(a) \wedge F(b) = F(a \wedge F(b))$.
4. For any $a \in R(Q)$, $F(a)$ is the least element of the set: $[a, 1] \cap F(R(Q))$.
5. For any $a \in R(Q)$, $a^* \leq F(a)$. In particular, the fixed points of F is a complete Heyting algebra and if a is fixed point of F then $a^* = a$.

The proofs of these claims are straightforward and are left to the reader. Notice, the involution for the fixed points of F is equal to the identity function. Clearly, we can ask for the following question: if Q is an idempotent, right-sided quantale and Q is involutive then what can we say about the binary operation &? The answer is as following proposition.

Proposition 9. *Let Q be an arbitrary idempotent and strictly right-sided quantale. If Q is involutive then Q is a complete Heyting algebra.*

Proof. Suppose Q is an idempotent and strictly right-sided quantale then for any elements a, b of Q, we know: $a\&b = a \wedge (1\&b)$. Now, if there exists an involution $* : Q \to Q$ then: $(a\&b)^* = b^*\&a^*$, for arbitrary elements $a, b \in Q$; taking $b = 1$ we have $a^* = (a\&1)^* = 1\&a^*$. In particular, if we take a^* we get $a^{**} = a = 1\&a$. Hence, for any $a \in Q$, $a = 1\&a$ and $a\&b = a \wedge b$. □

If Q is an idempotent quantale we can actually consider the morphism $j : Q \to Q$ given by: $j(a) = 1\&a\&1$. Clearly, j is a closure operator and the fixed points of j are strictly two-sided elements. In particular, the fixed points is not only a subquantale of Q it is a complete Heyting algebra.

Section 2

In this section we shall introduce the concept of an orthomodular lattice and we study the relation with idempotent and strictly right-sided quantales. In paticular, we want to see what is the relation between these kind of quantales and the algebraic foundations of quantum mechanics.

Definition 10. A complete lattice $L = (L, \vee, \wedge, \perp, 0, 1)$ is a complete orthocomplemented lattice if there exists a unary operation $\perp : L \to L$ satisfying the conditions:

1. $a^{\perp\perp} = a$;
2. $(\bigvee_{i\in I} a_i)^{\perp} = \bigwedge_{i\in I} a_i^{\perp}$;
3. $a \wedge a^{\perp} = 0$;
4. $a \vee a^{\perp} = 1$,

for all $a, a_i \in L$ and any set I.

Moreover, L is a complete orthomodular lattice if it satisfies the following weak modularity property:

Given any $a, b \in L$ with $a \leq b$ then $b = a \vee (a^{\perp} \wedge b)$ (equivalently, $a = (a \vee b^{\perp}) \wedge b$).

In [10] and [11] the first author and Beatriz-Rumbos study a non-commutative operation. Namely, the Sasaki projection $\&^F$; we shall treat the Sasaki projection as a binary connective. The definition of $\&^F$ is as follows: given any pair of elements a, b of an orthomodular lattice L, $a \&^F b = (a \vee b^{\perp}) \wedge b$. The reader can consult [9] for the properties of this binary connective. Also, until our knowledge, P.D. Finch in [5] was the first person who consider the Sasaki projection as a binary connective. As is well know the standard connectives used in classical logic or intuitionistic logic do not work very well for orthomodular lattices. We shall see that this connective $\&^F$ works very well for orthomodular lattices despite it is not an associative operation unless we have a Boolean algebra. We want to pointed out that in any orthomodular lattice, the Sasaki projection can be defined, no completeness assumption is required. On the other hand, quantales were defined from the beginning as a complete lattice.

We shall investigate now the connection between the operations $\&$ and $\&^F$ whenever $R(Q)$ is in fact an orthomodular lattice. We begin first with the following:

Example 11. Suppose S is a complete orthocomplemented lattice and the quantale $Q(S)$ of sup-preserving mappings from S to itself, with the supremum given by the pointwise ordering of mappings, with the multiplication & corresponding to composition of mappings and with the unit given by the identity mapping.

The involution $*$ on $Q(S)$ is defined by:

$$\varphi^*(s) = \left(\bigvee_{\varphi(t) \leq s^\perp} t \right)^\perp$$

for $\varphi \in Q(S)$ and $s, t \in S$.

The right-sided (left-sided) elements of $Q(S)$ are of the form:

$$\lambda_s(t) = \begin{cases} 0 & \text{if } t \leq s\,, \\ 1 & \text{if } t \not\leq s\,, \end{cases} \qquad \kappa_s(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ s & \text{if } t \neq 0. \end{cases}$$

Moreover, the involution and the negation operation satisfying the following rules:

i. $\lambda_s^* = \kappa_{s^\perp}$.

ii. $\lambda_s^\perp = \lambda_{s^\perp}$.

iii. ${}^\perp\kappa_s = \kappa_{s^\perp}$.

iv. $\lambda_s \& \lambda_s^* \& \lambda_s = \lambda_s$.

See again [9] for details (pp. 356–357). As a consequence of these properties, C.J. Mulvey and J. Pellettier proved the following proposition.

Proposition 12. *For any complete orthocomplemented lattice S, the mapping*

$$\chi \colon S \to R(Q(S))$$

defined by

$$\chi(s) = \lambda_{s^\perp}\,.$$

is an isomorphism of complete orthocomplemented lattices.

In fact, we can actually prove the following proposition.

Proposition 13. *If L is a complete orthomodular lattice then there is an isomorphism $\chi \colon L \to R(Q(L))$ given by the following rule:*

$$\chi(a) = \lambda_{a^\perp}.$$

Proof. We only need to show the orthomodular property; i.e., if $\lambda_s \leq \lambda_t$ then $\lambda_t = \lambda_s \vee (\lambda_{s^\perp} \wedge \lambda_t)$. Since L is in particular an ortholattice, and applying the proposition above, the morphism $\chi : S \to R(Q(S))$ preserves suprema, infima and complements. Hence, S is an isomorphism of complete orthomodular lattices. □

If Q is a Gelfand quantale and $a \in R(Q)$ then the element $1\&a$ is two-sided. In particular, for any element $b \in R(Q)$, $b\&a = b \wedge a$. We always have a morphism $G : Q \to Q(R(Q))$ given by: $G_a(b) = a^*\&b,$ for any $a, b \in Q$. C.J. Mulvey and J.W. Pelletier noticed in [9], G is a morphism of Gelfand quantales if and only if $R(Q)$ is a complete orthocomplemented lattice. Since in $R(Q)$ the two-sided elements satisfy: $a\&b = a \wedge b$ it is reasonable to assume $R(Q)$ is a complete orthomodular lattice; under this assumption we have the following result.

Lemma 14. *If Q is a Gelfand quantale and $R(Q)$ is a complete orthomodular lattice then for arbitrary elements $a, b \in R(Q)$ we have $b\&^F a \leq a\&b$. In particular, $a\&b = a \wedge (1\&b) = (1\&b)\&^F a$.*

Proof. $b\&^F a = a \wedge (b \vee a^\perp) \leq a^*\&(b \vee a^\perp) = (a^*\&b) \vee (a^*\&a^\perp) = a^*\&b \leq 1\&b$.

As $b\&^F a \leq a$, we have that $b\&^F a \leq a \wedge (1\&b) = a\&b$. □

By the previous results, if $R(Q)$ is an orthomodular lattice, we have the following inequalities for arbitrary elements a, b of $R(Q)$:

$$a \wedge b \leq b\&^F a \leq a\&b = a \wedge (1\&b) = (1\&b)\&^F a.$$

This result tells us not only the relation between the operations & and $\&^F$. The crucial thing here is that a and $1\&a$ are compatible elements. The definition of this concept is as follows.

Definition 15. Let L be an orthomodular lattice. We say $a, b \in L$ are compatible elements if and only if $b\&^F a = a \wedge b$.

For a discussion of this notion the reader can see for instance [2], Chapter 12.

We just mention that whenever a, b are compatible elements it is easy to see that if $b\&^F a = a \wedge b$ then $a\&^F b = a \wedge b$ also.

The simplest example of a pair of elements a, b which are compatible is whenever one of these elements belongs to the center $Z(L)$ of the orthomodular lattice L. The definition of the center is as follows.

Definition 16. Let L be an arbitrary orthomodular lattice. The center of L, denoted by $Z_F(L)$ is the set

$$Z_F(L) = \{a \in L | a\&^F b = b\&^F a = a \wedge b \quad \forall b \in L\}.$$

Notice that $Z_F(L)$ is a Boolean sublattice of L. If Q is a Gelfand quantale, we can define the center $Z(Q)$ of Q in the same way but using the & operation. If we assume that $R(Q)$ is an orthomodular lattice then we can consider also the center $Z_F(R(Q))$. Formally, we have the following definition.

Definition 17. Let Q be a Gelfand quantale such that $R(Q)$ is an orthomodular lattice then:

1. $Z(R(Q)) = \{a \in R(Q) | a\&b = b\&a \quad \forall b \in R(Q)\}$.
2. $B(Q) = R(Q) \cap L(Q)$.
3. $Z_F(R(Q)) = \{a \in R(Q) | a\&^F b = b\&^F a = a \wedge b \quad \forall b \in R(Q)\}$.

The next proposition discusses the relation between the centers.

Proposition 18. $Z(R(Q)) \subset B(Q) \subset Z_F(R(Q))$.

Proof. Suppose $a \in Z(R(Q))$ then for all $b \in R(Q)$, $a\&b = b\&a$. In particular $a\&1 = 1\&a$, so $a \in B(Q)$.

Now, suppose $a \in B(Q)$, then $a\&1 = 1\&a$, let $b \in R(Q)$, we have then

$$b \wedge a \leq a\&^F b \leq b\&a = b \wedge (1\&a) = b \wedge a.$$

So a is compatible with b and $b\&^F a = (b \vee a^{\perp}) \wedge a = b \wedge a = a\&^F b$. □

Proposition 19. *If L is an orthomodular lattice and $j : L \to L$ is a closure operation satisfying the Borceux-Van Den Bossche's conditions then L is an involutive and strict right-sided quantale.*

Proof. Just define $a\&b = a \wedge j(b)$. □

Given a complete orthomodular lattice L there is the notion of the central cover (denoted by $e(a)$) of an element a. The central cover allows us to give an example of a function satisfying the hypothesis of the last proposition.

Definition 20. Let L be a complete orthomodular lattice. For each $a \in L$, we define

$$e(a) = \wedge \{z \in Z_F(L) \mid a \leq z\}.$$

Proposition 21. *The map $e : L \to L$ satisfies the Borceux-Van Den Bossche conditions. L has a binary operation $\&$, defined by $a\&b = a \wedge e(b)$, making it an idempotent and strict right-sided quantale.*

Proof. Clearly, $1_L \leq e$ and $e^2 = e$.

Now, it is not hard to show: $e(a \wedge e(b)) = e(a) \wedge e(b)$ since $e(a \wedge z) = e(a) \wedge z$, whenever $z \in Z_F(L)$.

We check the last two properties.

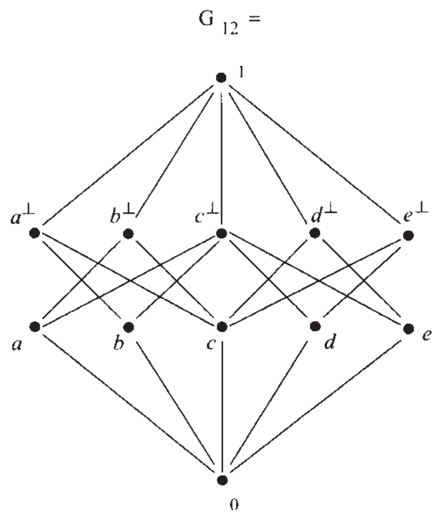
$$a \wedge (e(\vee_{i\in I} b_i)) = a \wedge (\vee_{i\in I} e(b_i)) = \vee_{i\in I} e(b_i) \&^F a = \vee_{i\in I}(e(b_i) \&^F a) = \vee_{i\in I}(a \wedge e(b_i)) \,.$$

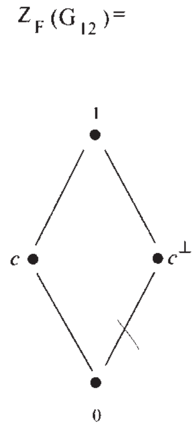
Since $e(b_i)$ is an element of the center of L for every $i \in I$ and $e(\vee_{i\in I} b_i) = \vee_{i\in I} e(b_i)$. Finally,

$$(\vee_{i\in I} a_i) \wedge e(b) = (\vee_{i\in I} a_i) \&^F e(b) = \vee_{i\in I}(a_i \&^F e(b)) = \vee_{i\in I}(a_i \wedge e(b)) \,.$$

Since $e(b)$ is an element of the center of L and $\&^F$ preserves suprema in the left variable. □

For more details about the central cover, the reader can see, for instance, [2]. We just mention two examples in this article. If we take $C(H)$ the center is trivial, i.e., $Z_F(C(H)) = \{0, 1\}$ therefore the quantale structure of $C(H)$ defined above is not really interesting. Clearly, not every orthomodular lattice has trivial center. Even finite orthomodular lattices have non trivial center. An orthomodular lattice L is irreducible if $Z_F(L) = \{0, 1\}$. As an example of an orthomodular lattice with non trivial center we just take G_{12}.





If L is a complete orthomodular lattice L and $j : L \to L$ is closure operator making L an idempotent and striclty right-sided quantale, we know $a\&b = a \wedge (1\&b)$; if L is in fact a Gelfand quantale then the involution is equal to the identity function and therefore L is a Boolean algebra.

There is another approach to study non-commutative binary operations. Namely, residuation theory. The reader can see [3] for a discussion of this theory. We shall introduce some basic facts about residuation theory.

Definition 22. Suppose A and B are two arbitrary posets. A morphism $f : A \to B$ is residuated if there is a morphism $f^+ : B \to A$ such that the following two conditions are satisfied:

If $a \in A$ and $b \in B$ are two arbitrary elements then:

$$a \leq f^+ \circ f(a) \text{ and } f \circ f^+(b) \leq b.$$

Usually, the morphism f^+ is called the residual of f; in categorical terms, we can say f is residuated if and only if f has a right adjoint.

Clearly, an arbitrary poset A has at least one residuated morphism. Namely, the identity morphism. We shall denote by $\text{Res}(A)$ the monoid of all residuated morphisms of a poset A; the operation $\circ$ is the composition of morphisms. Now, thanks to the work of D.J. Foulis and M.F. Janowitz, we can present some algebraic considerations we think are important for the algebraic foundations of quantum mechanics.

We shall present two examples of residuated endomorphisms.

Example 23. 1. Let X be a non-empty set, and let $P(X)$ be the Boolean Algebra of all subsets of X. If R is a relation of X consider the following endomorphism $\psi_R : P(X) \to P(X)$, described by:

$$\psi_R(A) = \{y \in X | \exists x \in A, xRy\}.$$

It is not hard to show that ψ_R is residuated with residual given by:

$$\psi_R^+ = i \circ \psi_{R^{op}} \circ i\,,$$

where $i : P(X) \to P(X)$ is the natural involution, sending any subset A of X to its complement and R^{op} denotes the converse relation of R.

Moreover, if $f : P(X) \to P(X)$ is a residuated map then $f = \psi_R$ for some relation R of X.

2. If V is a vector space and f is linear transformation on V then f induces a residuated mapping on the lattice of subspaces of V, namely the function given by: $M \longmapsto \{f(m)|m \in M\}$.

Notice the algebraic difference in the example of the relations of a set A. Clearly, we have a quantale if we allow in the definition of a quantale the operation & is just a residuated morphism. Hence, if we start with a complete lattice both definitions are equivalent.

As we said in the introduction, the notion of a quantifier on a Boolean Algebra is due to P. Halmos; latter M.F. Janowitz generalizes this idea for orthomodular lattices. We begin first with the following definition.

Definition 24. Let L be an orthomodular lattice. A function $F : L \to L$ will be called a quantifier if it satisfies:

1. $F(0) = 0$.
2. For any element a in L, we have: $a \leq F(a)$.
3. $F(a \wedge F(b)) = F(a) \wedge F(b)$, where a, b are arbitrary elements of L.

There are at least to special quantifiers on L. Namely, the discrete quantifier is just the identity on L and the indiscrete one is defined by: $F(0) = 0$ and $F(a) = 1$ if $a \neq 0$.

Moreover, in Example 22, ψ_R turns out to be a residuated quantifier iff R is an equivalence relation.

If Q is an idempotent, right-sided quantale then the function $j : Q \to Q$ given in Remark 1 is a residuated map. Also, if L is a complete orthomodular lattice, the central cover $e : L \to L$ is also an example of a residuated map.

Important consequences of this definition are contained in the next

Theorem 25. (Janowitz) *Let L be an arbitrary orthomodular lattice. If $F : L \to L$ is a quantifier, F satisfies:*

1. *$F(1) = 1$.*
2. *F is idempotent.*
3. *F preserves order.*
4. *$F(F(a)^{\perp}) = F(a)^{\perp}$.*
5. *F is a projection in $S(L)$.*
6. *The fixed points of F is a suborthomodular lattice of L.*
7. *F preserves arbitrary infima and suprema whenever they exist in L.*

There is also a simple consequence of this theorem. If L is a complete orthomodular lattice and F is a quantifier then for any element a of L the set $[a, 1] \cap F(L)$ has a least element. Namely – $F(a)$.

After the considerations we made in Section 1, we can work in a more general framework.

Definition 26. By bounded residuated semigroup with binary meets we understand a semigroup S with bounds $0, 1$ in such a way that the left multiplication and the right multiplication are residuated maps; i.e, the following holds:

1. Given any $a \in S$, the left multiplication $\lambda_a(b) = a \cdot b$ is residuated.
2. Given any $a \in S$, the right multiplication $\rho_a(b) = b \cdot a$ is residuated.

As a consequence we have the following lemma.

Lemma 27. *If $S = (S, \cdot, 0, 1, \wedge)$ is a bounded residuated semigroup with bounds $0, 1$ and binary meets satisfying:*

1. *$a \cdot a = a$.*
2. *$a \cdot 1 = a$.*
3. *$a \cdot 0 = 0 \cdot a = 0$.*

Then the following is true:

1. *For any $a, b \in S$, $\quad a \wedge b \leq a \cdot b$.*

2. *The function $F : S \to S$ given by $F(a) = 1 \cdot a$ is a quantifier and $a \cdot b = a \wedge (1 \cdot b)$.*

3. *If S has arbitrary suprema and the binary operation $\cdot$ preserves these suprema then F is a residuated quantifier.*

Proof. Since S is an idempotent semigroup we have:

$$(a \wedge b) \cdot (a \wedge b) \leq a \cdot a \wedge b \leq a \cdot b.$$

Now, from 1 we know: $a \wedge (1 \cdot b) \leq a \cdot (1 \cdot b) = (a \cdot 1) \cdot b = a \cdot b$.

Clearly, $F(0) = 0$, $a \leq F(a)$ and $F(a \wedge b) = F(a) \wedge F(b)$. Since, $b = b \cdot b \leq 1 \cdot b$ and $F(a) \wedge F(b) = 1 \cdot (a \cdot b)$, $F(a \wedge F(b) = 1 \cdot (a \cdot b)$.

The last claim follows easily. □

Clearly, any quantale is a bounded residuated semigroup with binary meets. We shall see now, what are the relations between idempotent right-sided quantales and the algebraic foundations of quantum mechanics.

A natural question is: Given an orthomodular lattice L can we define a binary operation & in such a way that L is a quantale? We shall see that this not possible if we want an idempotent, right-sided quantale. First of all, a quantifier F is weak if $F(0)$ is not neecessarily equal to zero. The next theorem proved by M.F. Janowitz in [7] tell us what are the weak quantifiers on the lattice $C(H)$ of the closed subspaces of a separable infinite dimensional Hilbert space, see [2] for more details on this hypothesis.

Theorem 28. (Janowitz) *Every weak quantifier in $C(H)$ is discrete or indiscrete.*

1. Given $A \in C(H)$ the mapping $F(B) = A$ if $B \leq A$, $F(A) = 1$ if the last claim do not hold, is an indiscrete quantifier; every indiscrete quantifier takes this form.

2. The mapping $F_A = B \vee A$ is a discrete weak quantifier if and only if A or $A^{\perp}$ is finite dimensional. If ψ is a discrete quantifier then $\psi = F_{\psi(0)}$.

Therefore, there is no way of inducing a on-trivial, non-commutative operation & in the lattice $C(H)$ making the last lattice an idempotent and strict right-sided quantale. We believe this is important since almost all the examples we know in quantale theory take care of this kind of quantales and we just saw that the only quantifiers for the closed subspaces of a Hilbert space are the trivial ones. Still a complete answer remains to be solve at least for the lattice of the closed subspaces of a Hilbert space. In other words, can we induce a non-trivial binary operation & in an orthomodular lattice L making L a quantale with certain properties?

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